Adequacy of matrices over commutative principal ideal domains

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Definition [Helmer, 1943]

Let R be a commutative ring with $1 \neq 0$ in which every finitely generated ideal is principal (Bézout ring). By a relatively prime part of $b \in R$ with respect to $a \in R$ written as $\mathrm{RP}(b, a)$, is a divisor t of $b = st$ such that $(t, a) = 1$, but $(s', a) \neq 1$ for any non-unit divisor $s' \in R$ of $s \in R$. The element s is called an adequate part of b with respect to a. The commutative ring R is called adequate if $\mathrm{RP}(b, a)$ exists for all $a, b \in R$ with $b \neq 0$.

This concept is basically a formalization of the properties of the entire analytic functions rings. Each commutative principal ideal domain (PID) is adequate, but the converse is not true. Each adequate ring is an elementary divisor ring. The ring of all continuous real-valued functions defined on a completely regular (Hausdorff) space X is an example of an adequate ring. Adequate rings with zero-divisors in their Jacobson radical were investigated by Kaplansky.

Bézout rings in which each regular element is adequate were investigated in Zabavsky and Gatalevych.

Definition [Gatalevych, 1998]

Let K be a Bézout ring and let $a \in K$. An element $b \in K$ is called adequate to $a \in K$ if the following conditions hold:

(i) there exist elements $s, t \in K$ such that $b = st$ and $tK + aK = K$;

(ii) $s'K + aK \neq K$ for each $s' \in K \setminus U(K)$ such that $sK \subset s'K \neq K$.

Matrices whose elements are relatively prime is called full matrices.

Theorem 1. [Gatalevych, Shchedryk, 2022]

Let R be an adequate ring in the sense of Gatalevych. The set of full nonsingular matrices from $M_2(R)$ is adequate in $M_2(R)$.

Theorem 2. [Gatalevych, Shchedryk, 2022]

Let R be an elementary divisor domain. The set of full singular matrices from $M_2(R)$ is adequate in the sense of Gatalevych in the set of full matrices in $M_2(R)$.

Let R be a commutative PID, and let $a, d, c \in R \setminus \{U(R) \cup \{0\}\}$ be pairwise relatively prime indecomposable elements. Let

$$A := \mathrm{diag}(a, a^2 dc); \qquad B := \begin{bmatrix} 1 & 0 \\ d & d^3c^2 \end{bmatrix}.$$

It is easy to check that both decompositions

$$B = ST = \underbrace{\mathrm{diag}(1, d)}_{S} \cdot \underbrace{\begin{bmatrix} 1 & 0 \\ 1 & d^2c^2 \end{bmatrix}}_{T};$$

$$B = S_1 T_1 = \underbrace{\begin{bmatrix} 1 & 0 \\ d^3 + d & d^3 \end{bmatrix}}_{S_1} \cdot \underbrace{\begin{bmatrix} 1 & 0 \\ -1 & c^2 \end{bmatrix}}_{T_1}.$$

satisfies the conditions of the Gatalevych's definition.

However, S is the left nontrivial divisor of S_1, because

$$S_1 = S \begin{bmatrix} 1 & 0 \\ 1 & d^2 \end{bmatrix}.$$

This means that in the decomposition of $B = ST$ we can inflate the matrix S to the matrix S_1 so that the new decomposition

$$B = S_1 T_1$$

also satisfies the Gatalevych's definition.

Let $U \in \mathrm{GL}_2(R)$. In the decomposition $B = (SU)(U^{-1}T)$ the matrix SU also satisfy the part (i) of the Gatalevych's definition but $U^{-1}T$ does not necessarily satisfy the part (ii). Indeed, if

$$A := \mathrm{diag}(a, a^2dc), \quad B := \underbrace{\mathrm{diag}(1, d)}_{S} \cdot \underbrace{\begin{bmatrix} 1 & 0 \\ 1 & d^2c^2 \end{bmatrix}}_{T}, \quad U := \begin{bmatrix} 1 & 0 \\ 1-d & 1 \end{bmatrix}.$$

Then $B = ST$ satisfy the Gatalevych's definition, but the decomposition $B = (SU)(U^{-1}T)$, where

$$U^{-1}T = \begin{bmatrix} 1 & 0 \\ d & d^2c^2 \end{bmatrix}$$

does not satisfy the part (ii) because

$$(A, U^{-1}T)_l = \mathrm{diag}(1, d) \neq I.$$

Together with V. Shchedryk we propose the following:

Definition [Bovdi-Shchedryk, 2023]

Let K be a Bézout ring (not necessary commutative) with $1 \neq 0$ and let $a \in K$. An element $b \in K$ is called a left adequate to $a \in K$ if there exist $s, t \in K \setminus U(K)$ such that $b = st$ and the following conditions hold:

(i) $s'K + aK \neq K$ for each $s' \in K \setminus U(K)$ such that $sK \subset s'K \neq K$ (i.e. each left divisor $s' \in K \setminus U(K)$ of s has a left nontrivial common divisor with the element a);

(ii) for each $t' \in K \setminus U(K)$ such that $tK \subset t'K \neq K$ there exists a decomposition $st' = pq$ such that $pK + aK = K$ (i.e. for each left divisor $t' \in K \setminus U(K)$ of t, the element st' has always a decomposition $st' = pq$ such that a and p already are left relatively primes).

The element s is called a left adequate part of b with respect to a. The right adequate part of b with respect to a is defined by analogy.

A subset $A \subseteq K$ is called left (right) adequate if each of its elements is left (right) adequate to all elements of A. If each element of A is left and right adequate to the rest of elements, then the set A is called adequate.

It is easy to see if K is a commutative PID, then our definition coincides with the definition given by Helmer.

Theorem 3. [Bovdi Shchedryk, 2023]

Let R be a commutative principal ideal domain such that $1 \neq 0$. The set of nonsingular 2×2 matrices over R is an adequate set.

Theorem 4. [Bovdi Shchedryk, 2023]

Let R be a commutative principal ideal domain such that $1 \neq 0$. Let A, B be nonsingular matrices in $M_2(R)$. If $B = ST$, where S is an adequate part of B with respect to A, then the matrix S is determined up to equivalence of matrices.

Bibliography

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