Noncommutative tensor triangulated categories and coherent frames

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Frames and topological spaces: adjoint functors

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- set of points of any frame form a topological space with open sets of the form $\Omega(u) = \{x : F \longrightarrow \{0,1\} \mid x(u) = 1\}$ for any $u \in F$:

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- an element a of a frame F is *finite* if for every subset $S \subseteq F$ with $a \leq \bigvee_{s \in S} s$, $\exists$ a finite subset $S' \subseteq S$ with $a \leq \bigvee_{s \in S'} s$. A frame is called *coherent* if every element of the frame can be expressed as a join of finite elements and the finite elements form a sublattice.

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$$\text{Spectral spaces} \equiv \text{Coherent frames}$$

Hochster dual

For a spectral space X,

- Hochster considered a new topology on X by taking as basic open subsets the closed sets with quasi-compact complements.
- The space so obtained called *Hochster dual* of X and denoted by $X^\vee$.
- It is also a spectral space.

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- However, Balmer showed that using subsets of $Spec(\mathbf{T})$, one can always classify objects of $\mathbf{T}$ modulo the basic operation: : cones, direct summands and tensor product.
- Precisely, "Thomason subsets" of $Spec(\mathbf{T}) \leftrightarrow$ "radical thick $\otimes$ ideals" of $\mathbf{T}$.

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- **Question**: Can Koch and Pitsch's point free approach be used in this noncomm. setup? Will it simplify the classification?
- We show that it is possible under an assumption which is satisfied by a large class of non-comm. TT categories.

Noncomm. Balmer's spectrum(Nakano, Vashaw, Yakimov)

- *triangulated subcat* $\mathbf{K} \subseteq \mathbf{T}$: for every $a \longrightarrow b \longrightarrow c \longrightarrow \Sigma a$ in $\mathbf{T}$, if two out of a, b, c belongs to $\mathbf{K}$, so does the third.

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$$\mathbf{I} \otimes \mathbf{J} \subseteq \mathbf{P} \implies \mathbf{I} \subseteq \mathbf{P} \text{ or } \mathbf{J} \subseteq \mathbf{P}$$

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- The noncomm. NVY spectrum $Spc(\mathbf{K})$: collection of prime ideals of $\mathbf{K}$ endowed with Zariski-like topology given by closed sets of the form

  $$V(S) = \{\mathbf{P} \in Spc(\mathbf{K}) \mid \mathbf{P} \cap S = \emptyset\}$$

  for all subsets S of $\mathbf{K}$.

Noncomm. support datum and universal property

Let $\mathcal{X}_{cl}(X)$ denote the collection of all closed subsets of a topological space X.

Definition (Nakano, Vashaw, Yakimov)

A (noncommutative) support datum on $\mathbf{T}$ is a pair (X, σ) where X is a top space and σ is a map $\mathbf{T} \longrightarrow \mathcal{X}_{cl}(X)$ s.t:

(1) $\sigma(0) = \emptyset$ and $\sigma(1) = X$

(2) $\sigma(a \oplus b) = \sigma(a) \cup \sigma(b), \quad \forall a, b \in Ob(\mathbf{T})$

(3) $\sigma(\sum a) = \sigma(a), \quad \forall a \in Ob(\mathbf{T})$

(4) If $a \longrightarrow B \longrightarrow c \longrightarrow \sum a$ is a distinguished triangle, then $\sigma(a) \subseteq \sigma(b) \cup \sigma(c)$

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- $V : \mathbf{T} \longrightarrow \mathcal{X}_{cl}(Spc(\mathbf{T})),\ a \mapsto supp(a)$ gives a support datum.

Theorem (Nakano, Vashaw, Yakimov)

The support V is final among all the support data σ of $\mathbf{K}$ such that $\sigma(A)$ is closed for each $A \in Ob(\mathbf{T})$.

Radical ideals form a frame

Borrowing idea from noncomm. ring theory, we defined

Definition (−, Mallick)

The *radical closure* of a thick tensor ideal $\mathbf{I}$ of a noncomm. tt-category $\mathbf{K}$:

$$\sqrt{\mathbf{I}} := \bigcap_{\mathbf{I} \subseteq \mathbf{P}} \mathbf{P}$$

where $\mathbf{P}$ denotes prime ideals of $\mathbf{K}$. If $\mathbf{I}$ is s.t $\mathbf{I} = \sqrt{\mathbf{I}}$, we call $\mathbf{I}$ radical.

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Proposition (−, Mallick)

Let $\mathbf{Rad_K}$ *denote poset of radical ideals of a noncomm. tt-category* $\mathbf{K}$ *satisfying* **Assumption**. *Then,* $\mathbf{Rad_K}$ *is a frame with following meet and join operations:*

$$\mathbf{I}_1 \bigwedge \mathbf{I}_2 := \mathbf{I}_1 \bigcap \mathbf{I}_2 \qquad \bigvee_{j \in J} \mathbf{I}_j := \sqrt{\bigcup_{j \in J} \mathbf{I}_j}$$

Radical ideals form a coherent frame

Let S be a set of objects in a noncomm. tt-category $\mathbf{K}$. We define $G(S)$ to be the set of objects of the forms:

(1) an iterated suspension or desuspension of an object in S,

(2) or a finite sum of objects in S,

(3) or objects of the form $s \otimes t$ and $t \otimes s$ with $s \in S$ and $t \in \mathbf{K}$,

(4) or an extension of two objects in S,

(5) or a direct summand of an object in S.

If $\mathbf{I}$ is a thick tensor ideal containing S, then clearly $G(S) \subseteq \mathbf{I}$. Hence, by induction, $G^{\omega}(S) := \bigcup_{n \in \mathbb{N}} G^n(S) \subseteq \mathbf{I}$. It may be easily verified that $G^{\omega}(S)$ is itself a thick tensor ideal and therefore it is the smallest thick tensor ideal containing S. We will denote it by $\langle S \rangle$.

Radical ideals form a coherent frame

Let S be a set of objects in a noncomm. tt-category $\mathbf{K}$. We define $G(S)$ to be the set of objects of the forms:

(1) an iterated suspension or desuspension of an object in S,

(2) or a finite sum of objects in S,

(3) or objects of the form $s \otimes t$ and $t \otimes s$ with $s \in S$ and $t \in \mathbf{K}$,

(4) or an extension of two objects in S,

(5) or a direct summand of an object in S.

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Proposition (−, Mallick)

Let $\mathbf{I}$ be a thick $\otimes$-ideal of $\mathbf{K}$. Then, $\sqrt{\mathbf{I}} = \langle \{k \in \mathbf{K} \mid k^{\otimes n} \in \mathbf{I} \text{ for some } n \in \mathbb{N}\} \rangle$.

Radical ideals form a coherent frame (cont.)

Theorem (−, Mallick)

The poset of radical ideals $\mathbf{Rad}_{\mathbf{K}}$ *of a noncomm. tt-category* $\mathbf{K}$ *satisfying* **Assumption** *forms a coherent frame.*

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Zariski spectrum: $Spec_{\mathbf{Rad}}(\mathbf{K})$ the spectral space associated to $\mathbf{Rad}_{\mathbf{K}}$

Classification: $Spc(\mathbf{K})$ and radical ideals

Theorem (−, Mallick)

Let $\mathbf{K}$ *be a noncomm. tt-category satisfying* **Assumption**. *Then,*

(1) *the frame-theoretic points of* $\mathbf{Rad}_{\mathbf{K}}$ *correspond bijectively to prime thick tensor ideals in* $\mathbf{K}$.

(2) *Under the above correspondence, a finite element* $\sqrt{k}$ *of* $\mathbf{Rad}_{\mathbf{K}}$ *corresponds to the set of prime thick tensor ideals* $\{\mathbf{P} \in Spc(\mathbf{K}) \mid k \notin \mathbf{P}\}$.

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Corollary (−, Mallick)

Let $\mathbf{K}$ *be a noncomm. tt-category satisfying* **Assumption**. *The noncomm. Balmer's spectrum* $Spc(\mathbf{K})$ *of* $\mathbf{K}$ *is the Hochster dual of the Zariski spectrum* $Spec_{\mathbf{Rad}}(\mathbf{K})$.

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Support and universal property

we introduce a notion of support for a noncomm. tt-category:

Definition (−, Mallick)

A support on $\mathbf{K}$ is a pair (F, d) where F is a frame and $d : Ob(\mathbf{K}) \longrightarrow F$ is a map satisfying:

(1) $d(0) = 0$ and $d(\mathbf{1}) = 1$

(2) $d(\sum k) = d(k) \ \ \forall k \in \mathbf{K}$

(3) $d(k \oplus t) = d(k) \vee d(t) \ \ \forall k, t \in \mathbf{K}$

(4) $d(k \otimes t) = d(k) \wedge d(t) = d(t \otimes k) \ \ \forall k, t \in \mathbf{K}$

(5) If $k \longrightarrow t \longrightarrow r \longrightarrow \sum k$ is a triangle in $\mathbf{K}$, then $d(t) \leq d(k) \vee d(r)$

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Theorem (–, Mallick)

Let $\mathbf{K}$ *be a noncomm. tt-category satisfying* **Assumption**.

- *Then the assignment* $s : Ob(\mathbf{K}) \longrightarrow \mathbf{Rad}_{\mathbf{K}}, \ \ k \mapsto \sqrt{k}$ *is a support. Moreover, it is initial among all supports.*
- *From this frame theoretic support datum, one can reconstruct the support datum on* $Spc(\mathbf{K})$ *as described by NVY.*

Nullstellensatz-like result

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Theorem (–, Mallick)

Let $\mathbf{K}$ *be a noncomm. tt-category satisfying* **Assumption**. *The following spaces are spectral and there is a homeomorphism between them:*

(1) *The frame* $\mathbf{Rad_K}$ *of radical thick tensor ideals of* $\mathbf{K}$ *endowed with the topology generated by the open sets*

$$\{\mathbf{I} \in \mathbf{Rad_K} \mid k \notin \mathbf{I}\} \qquad \forall\ k \in \mathbf{K}. \tag{1}$$

(2) *The poset* $\Omega(Spc(\mathbf{K})^\vee)$ *of open subsets of* $Spc(\mathbf{K})^\vee$ *(or equivalently, open subsets of* $Spec_{Zar}(\mathbf{K})$*) endowed with the topology generated by the open sets*

$$\{V \in \Omega(Spc(\mathbf{K})^\vee) \mid V \not\supseteq U\} \qquad \forall\ U \in \Omega(Spc(\mathbf{K})^\vee). \tag{2}$$

Question:

Can we remove the **Assumption**?

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Thank You!