Powers of irreducibles in rings of integer-valued polynomials

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Happy Birthday

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Factorizations and $\mathrm{Int}(\mathbb{Z})$

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$$2 \cdot 3 \cdot 7 \cdot \binom{x}{42} = \binom{x}{41} \cdot (x-41)$$

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How to recognize **?**

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$\binom{x}{n}$ is irreducible in $\mathrm{Int}(\mathbb{Z})$ for all $n \in \mathbb{N}_0$.

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Theorem (R., Windisch; 2021)

$\binom{x}{n}$ is **absolutely** irreducible in $\mathrm{Int}(\mathbb{Z})$ for all $n \in \mathbb{N}_0$.

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$$\binom{x}{3}^k = f \cdot g \quad \Longrightarrow \quad f = \pm \left(\frac{x}{3}\right)^{k_0} \left(\frac{x-1}{2}\right)^{k_1} \left(\frac{x-2}{1}\right)^{k_2}$$

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$$\begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} k_0 \\ k_1 \\ k_2 \end{pmatrix} \geq 0$$

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$$\Longrightarrow \quad f = \binom{x}{3}^{k_0}$$

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$$\binom{x}{n} = \frac{x(x-1)\cdots(x-n+1)}{n!} \in \operatorname{Int}(\mathbb{Z}) \quad \text{abs. irred.}$$

Split i.-v. polynomials

$$h = \frac{(x-a_1)^{k_1}(x-a_2)^{k_2}(x-a_3)^{k_3}\cdots(x-a_n)^{k_n}}{m}$$

Split i.-v. polynomials

$$h^{k} = \left(\frac{(x-a_1)^{k_1}(x-a_2)^{k_2}(x-a_3)^{k_3}\cdots(x-a_n)^{k_n}}{m}\right)^{k}$$

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Split i.-v. polynomials

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p^e

Split i.-v. polynomials over a DVR D

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Split polynomials

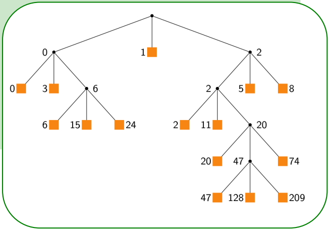
Theorem (Frisch, Nakato, R., 2022)

Let R be a discrete valuation domain with finite residue field, $S \subseteq R$ finite. Then

$$\frac{h_0}{p^e} \quad \text{with} \quad h_0 = \prod_{s \in S} (x - s)^{k_s}$$

is absolutely irreducible if and only if

- S is a balanced set
- h_0 is the equalizing polynomial of S
- p^e is the fixed divisor of h_0



All int.-val. polynomials over DVRs

$$h = \frac{f_1^{k_1} f_2^{k_2} f_3^{k_3} \cdots f_r^{k_r}}{p^n}$$

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All int.-val. polynomials over DVRs

$$\boxed{\begin{array}{c} f \sim h^r \\ g \sim h^s \end{array}}\ \textbf{?}$$

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int.-val.

Ex. in $\operatorname{Int}\left(\mathbb{Z}_{(3)}\right)$

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$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{pmatrix} \in \frac{\overset{2\cdot 3}{\ell}}{2}\begin{pmatrix} 1 \\ 3 \\ 1 \\ 1 \end{pmatrix} + \underbrace{\ker \begin{array}{c} \begin{array}{cccc} x^2+9 & x-5 & x-1 & x-7 \end{array} \\ \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \end{array} \begin{array}{l} x=0 \\ x=4 \end{array}}_{(0,-1,0,0)^t}$$

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$\mathrm{Int}(\mathbb{Z}_{(3)})$

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- F^S factors uniquely $S = 2(n+1)n^{q^{\lceil \frac{n}{2} \rceil}}$ where $q = |R/pR|$

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- What are the limits of the non-uniqueness of these factorizations?