# On some combinatorial invariants associated with commutative rings

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# Graphs associated with commutative rings

- The zero-divisor graph $G$ of a commutative ring $R$ with unity was introduced by Beck [1]. The graph $G$ is defined to be the graph with vertex set as $R$, and two distinct vertices $x, y \in R$ are adjacent in $G$ if and only if $xy = 0$ in $R$. The main objective of the investigation of associating a graph to $R$ is to study the interplay between combinatorial and ring theoretic properties of $R$. By the definition of Beck's zero-divisor graph $G$, it appears that the vertex 0 in $G$ is adjacent to every other vertex of $G$ and other adjacencies in $G$ are between zero-divisors of $R$. Therefore, $G$ is a simple connected graph, that is, $G$ is without parallel edges and self loops.

- Let $Aut(G)$ denotes the automorphism group of a graph $G$, consider the group action $Aut(G)$ *acting on* $V(G)$ by some permutation of $Aut(G)$, that is, $Aut(G) \times V(G) \to V(G)$ is given as, $\sigma(v) = u$, where $\sigma \in Aut(G)$ and $v, u \in V(G)$ are any two vertices of $G$.
- Let $n = p^{\alpha}$, where $p$ is a prime number and $\alpha \geq 1$. If $R \cong \mathbb{Z}_{p^{\alpha}}$, then orbits of the group action $Aut(G) \times R \longrightarrow R$ are given as, $\mathcal{O}_{\alpha,p^i} = \{p^i b (\text{mod } p^{\alpha}) \mid b \in \mathbb{Z}, (b, p) = 1\}$, $\mathcal{O}_{\alpha,p^i}$ is the $p^i$-th orbit of $R$, where $i \in [0, \alpha - 1]$.

- It can be verified that for $i \in [0, \alpha]$, the size of sets $\mathcal{O}_{\alpha, p^i} = \phi(\frac{p^\alpha}{p^i})$ , where $\phi$ is the Euler's totient function.
- Note that $1, p, \ldots, p^\alpha$ are the representatives of orbits $\mathcal{O}_{\alpha,1}, \mathcal{O}_{\alpha,p}, \ldots, \mathcal{O}_{\alpha,p^\alpha}$ and divisors of the order of $R$.
- There is an advantage for knowing orbits of the group action $Aut(G) \times R \to R$, since we get some structural information about some elements of $R$ from orbits of the group action. As a consequence, we do not consider all elements of $R$ to decode the symmetry of $G$. We explore this information to reveal some interesting spectral properties of the combinatorial structure $G$.

- The following example which illustrates the zero-divisor graph associated with $\mathbb{Z}_{3^3}$ with its orbits.
- Let $R = \mathbb{Z}_{3^3}$ and let $G$ (shown in Figure below) be a graph associated with $R$. Consider the group action $Aut(G) \times \mathbb{Z}_{3^3} \to \mathbb{Z}_{3^3}$. The orbits of this action are: $\mathcal{O}_{3,1} = \{1, 2, 4, \cdots, 25, 26\}$, $\mathcal{O}_{3,3} = \{3, 6, 12, 15, 21, 24\}$, $\mathcal{O}_{3,3^2} = \{9, 18\}$ and $\mathcal{O}_{3,3^3} = \{0\}$.

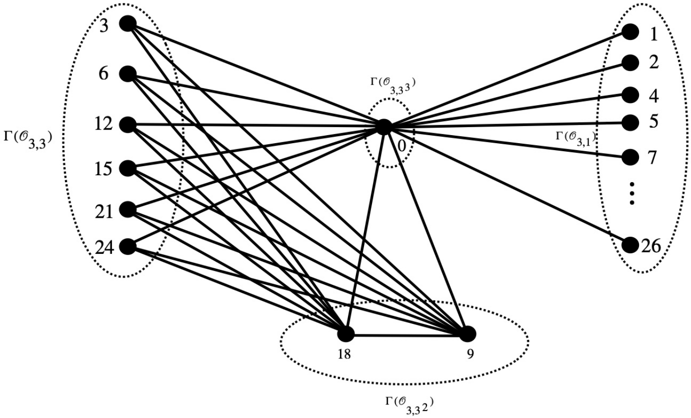



Figure: Graph associated with $\mathbb{Z}_{3^3}$

- The subgraphs $\Gamma(\mathcal{O}_{3,1})$, $\Gamma(\mathcal{O}_{3,3})$, $\Gamma(\mathcal{O}_{3,3^2})$ and $\Gamma(\mathcal{O}_{3,3^3})$ induced in $G$ by orbits of the group action as shown in the Figure above are : $\Gamma(\mathcal{O}_{3,1}) = \overline{K}_{18}$, $\Gamma(\mathcal{O}_{3,3}) = \overline{K}_6$, $\Gamma(\mathcal{O}_{3,3^2}) = K_2$, and $\Gamma(\mathcal{O}_{3,3^3}) = K_1$. Here $\overline{K}_n$ denotes the compliment of a complete graph $K_n$.
- Observe that the graph $G$ is a *join* of independent (no two vertices are adjacent) and complete (every pair of vertices are adjacent) parts of the graph. This type of division of vertices in $G$ allows us to discuss some interesting combinatorial properties of commutative rings.

# Young diagrams associated with commutative rings

- **Threshold graph:** A graph $\Gamma$ is called a *threshold graph* if it is generated by a binary code of the type $(b_1, b_2, \cdots, b_n)$, where $b_i = 0$ if vertex $v_i$ is being added as an isolated vertex in $\Gamma$ and $b_i = 1$ if $v_i$ is being added as a dominating vertex in $\Gamma$. In fact $\Gamma$ is threshold if and only if it can be generated by a binary code of the type $0^{s_1}1^{t_1}0^{s_2}1^{t_2}\dots 0^{s_k}1^{t_k}$, where powers of bits $s_i, t_i$, $1 \leq i \leq k$, are positive integers.
- Several invariants related to $\Gamma$ have been determined using the properties of powers $s_i$ and $t_i$ of bits 0 and 1 in [7].

- Moreover the powers of bits 0 and 1 can be used to establish the nullity, multiplicity of some non-zero eigenvalues and the Laplacian eigenvalues of $\Gamma$. Note that the Laplacian eigenvalues of $\Gamma$ are eigenvalues of a matrix $D(\Gamma) - A(\Gamma)$, where $D(\Gamma)$ is a diagonal matrix of vertex degrees and $A(\Gamma)$ is the familiar $(0, 1)$ adjacency matrix of $\Gamma$.
- The authors in [3] confirmed that the graph realized by a finite abelian $p$-group of rank 1 is a threshold graph. In fact, they proved the following intriguing result for a finite abelain $p$-groups of rank 1.

**Theorem 2 [3].** If $\mathcal{G}$ is a finite abelian $p$-group of rank 1, then the graph $\Gamma(\mathcal{G})$ realized by $\mathcal{G}$ is a threshold graph.

- We continued the investigation that was initiated in [3] and considered graphs discussed in Section 1. It was shown in [6] that graphs associated with a finite field, $\mathbb{F}_q$, ring of the type $\mathbb{Z}_2 \times \mathbb{F}_q$ and polynomial rings of types $\mathbb{Z}_p[x]/(x^p)$, $\mathbb{Z}_{p^\alpha}[x]/(x^2, px)$, $\mathbb{Z}_p[x, y]/(x^3, xy, y^2)$, $\mathbb{Z}_{p^2}[x]/(x^2 - p, px)$ are threshold.

- The *degree sequence* of a graph $\Gamma$ is given by $\pi(\Gamma) = (d_1, d_2, \cdots, d_n)$, which is the non-increasing sequence of non-zero degrees of vertices of $\Gamma$.
- For a graph $\Gamma$ of order $n$ and size $m$, let $d = [d_1, d_2, \cdots, d_n]$ be a sequence of non-negative integers arranged in non-increasing order, which we refer to as a partition of $2m$. Define the transpose of the partition as $d^* = [d_1^*, d_2^*, \cdots, d_r^*]$, where $d_j^* = |\{d_i : d_i \geq j\}|$, $j = 1, 2, \cdots, r$. Therefore $d_j^*$ is the number of $d_i$'s that are greater than equal to $j$. A sequence $d^*$ is called the conjugate sequence of $d$.

- The another interpretation of a degree sequence is the (Young diagram) denoted by $Y(d)$ corresponding to $d_1, d_2, \cdots, d_n$ consists of $n$ left justified rows of boxes, where the $i^{th}$ row consists of $d_i$ boxes (blocks), $i = 1, 2, \cdots, n$. Note that $d_i^*$ is the number of boxes in the $i^{th}$ column of the Young diagram with $i = 1, 2, \cdots, r$.
- If $d$ represents the degree sequence of a graph, then the number of boxes in the $i^{th}$ row of the Young diagram is the degree of vertex $i$, while the number of boxes in the $i^{th}$ row of the Young diagram of the transpose is the number of vertices with degree at least $i$. The *trace* of a Young diagram $tr(Y(d))$ is, $tr(Y(d)) = |\{i : d_i \geq i\}| = tr(Y(d^*))$, which is the length of "diagonal" of the Young diagram for $d$ (or $d^*$).

- The degree sequence is a graph invariant, so two isomorphic graphs have the same degree sequence. In general, the degree sequence does not uniquely determine a graph, that is, two non-isomorphic graphs can have the same degree sequence. However, for threshold graphs, we have the following result.
- **Proposition 2 [5].** Let $\Gamma_1$ and $\Gamma_2$ be two threshold graphs and let $\pi_1(\Gamma_1)$ and $\pi_2(\Gamma_2)$ be degree sequences of $\Gamma_1$ and $\Gamma_2$ respectively. If $\pi_1(\Gamma_1) = \pi_2(\Gamma_2)$, then $\Gamma_1 \cong \Gamma_2$.

- The Laplacian spectrum of threshold graphs $\Gamma$, which we denote by $\ell - spec(\Gamma)$, has been studied in [2, 4]. In [2], the formulas for the Laplacian spectrum, the Laplacian polynomial, and the number of spanning trees of a threshold graph are given. It is shown that the degree sequence of a threshold graph and the sequence of eigenvalues of its Laplacian matrix are "almost the same" and on this basis, formulas are given to express the Laplacian polynomial and the number of spanning trees of a threshold graph in terms of its degree sequence.
- The following is the fascinating result regarding the Laplacian eigenvalues of graphs associated with polynomial rings of types $\mathbb{Z}_p[x]/(x^p)$, $\mathbb{Z}_{p^\alpha}[x]/(x^2, px)$, $\mathbb{Z}_p[x, y]/(x^3, xy, y^2)$, $\mathbb{Z}_{p^2}[x]/(x^2 - p, px)$.

**Theorem.** The Laplacian eigenvalues of graphs associated with polynomial rings of types $\mathbb{Z}_p[x]/(x^p)$, $\mathbb{Z}_{p^\alpha}[x]/(x^2, px)$, $\mathbb{Z}_p[x, y]/(x^3, xy, y^2)$, $\mathbb{Z}_{p^2}[x]/(x^2 - p, px)$ are representatives of orbits of the group action $Aut(G) \times V(G) \longrightarrow V(G)$, where $G$ is any graph associated with above polynomial rings. In particular, for a ring of the type $\mathbb{Z}_{p^\alpha}$, the representative $1, p, p^2, \cdots, p^{k-1}, p^k$ of orbits $\{\mathcal{O}_{k,p^k}\} \cup \{\mathcal{O}_{k,p^i} : 0 \leq i \leq k-1\}$ are the Laplacian eigenvalues of $G$ associated with $\mathbb{Z}_{p^\alpha}$, that is, $\ell - spec(G) = \{0, 1, p, p^2, \cdots, p^{k-1}, p^k\}$.

## Example

- For a ring $R = \mathbb{Z}_2^4$ , the degree sequence $\sigma$ of $G$ associated with $R$ is,

$$\sigma = (15, 7, 3, 3, 2, 2, 2, 2, 1, 1, 1, 1, 1, 1, 1, 1).$$

The conjugate sequence of $\sigma$ is,

$$\sigma^* = (2^4, 2^3, 2^2, 2, 2, 2, 2, 1, 1, 1, 1, 1, 1, 1, 1).$$

The elements of $\sigma^*$ are representatives of orbits of the group action $Aut(G) \times R \longrightarrow R$, where $G$ is the graph associated with $R$.

- If $a = (a_1, a_2, \cdots, a_r)$ and $b = (b_1, b_2, \cdots, b_s)$ are non-increasing sequences of real numbers. Then $b$ *weakly majorizes* $a$, written as $b \succeq a$, if $r \geq s$,

$$\sum_{i=1}^{k} b_i \geq \sum_{i=1}^{k} a_i,$$

where $1 \leq k \leq s$, and

$$\sum_{i=1}^{r} b_i \geq \sum_{i=1}^{s} a_i. \qquad (1)$$

If $b$ weakly majorizes $a$ and equality holds in (1), then $b$ *majorizes* $a$, written as $b \succ a$.

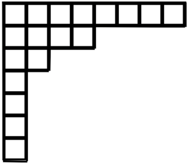
- Let $\pi_1, \pi_2, \cdots, \pi_n \in \mathbb{Z}_{>0}$ and $\pi_1^\bullet, \pi_2^\bullet, \cdots, \pi_n^\bullet \in \mathbb{Z}_{>0}$ be some partitions of $n \in \mathbb{Z}_{>0}$. A partition of eigenvalues $\pi = (\pi_1, \pi_2, \cdots, \pi_n)$ of a graph $\Gamma$ is said to be a *threshold eigenvalues partition* if $\pi_i = \pi_i^\bullet + 2$ for all $i$ with $1 \leq i \leq tr(Y(\pi))$, where $tr(Y(\pi))$ denotes trace of a Young diagram $Y(\pi)$.
- Here is an example which illustrates that the threshold eigenvalues partition of a graph realized by some graph associated with a finite ring $R_1$ majorizes the degree partition of the graph realized by some other finite ring $R_2$.

- Let $R_1 = \mathbb{Z}_2^3$ and $R_2 = \mathbb{Z}_3^2$ be two rings. The degree partitions $\pi_1^\bullet$ and $\pi_2^\bullet$ of graphs associated with rings $R_1$ and $R_2$ are listed below as,

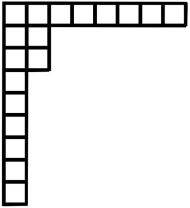
$$\pi_1^\bullet = (7, 3, 2, 2, 1, 1, 1, 1),$$
$$\pi_2^\bullet = (8, 2, 2, 1, 1, 1, 1, 1, 1).$$

- The partitions $\pi_1^\bullet, \pi_2^\bullet \in \mathcal{P}(18)$, where $\mathcal{P}(18)$ is the set of all partitions of 18. The partition $\pi_1 = (8, 4, 2, 1, 1, 1, 1)$ is the threshold eigenvalues partition of $\Gamma(G_1)$. The Young diagrams of partitions $\pi_1$ and $\pi_2^\bullet$ are shown in the Figure below.

$\pi_1$ :



$\pi_2$ :



# On Stanley's Conjecture

- Let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_r)$ be a partition of $n \in \mathbb{Z}_{>0}$ denoted by $\lambda \vdash n$.
- Let $\pi$ and $\sigma$ be two degree sequences of graphs realized by finite abelian $p$-groups of rank 1 such that $\pi, \sigma \vdash m$, where $m \in \mathbb{Z}_{>0}$. Then $\pi \succ \sigma$ if and only if $Y(\pi)$ can be obtained from $Y(\sigma)$ by moving blocks of the highest row in $Y(\sigma)$ to lower numbered rows.

- Young's partition lattice $L(m, n)$ is defined to be the poset of integer partitions $\lambda = (0 \leq \lambda_1 \leq \cdots \leq \lambda_m \leq n)$ equipped with the partial order, $\lambda = (0 \leq \lambda_1 \leq \cdots \leq \lambda_m)$"$\leq$" $\mu = (\mu_1 \leq \cdots \leq \mu_m)$ if and only if $\lambda_i \leq \mu_i$ for all $1 \leq i \leq m$. We can visualize the elements of this poset as Young diagrams that "fit in", ordered by inclusion as shown in the Figure below. Note that $L(m, n)$ is a ranked poset, where the rank of $\lambda$ is given by $\lambda_1 + \cdots + \lambda_m$.

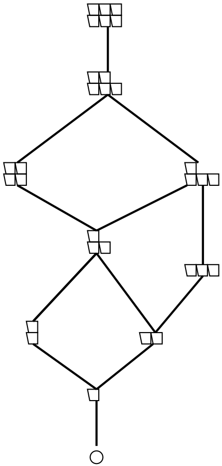


Figure: The Hasse Diagram $L(2, 3)$

- A decomposition of a poset $\mathcal{P}$ is a partition of $\mathcal{P}$ into disjoint subsets $\mathcal{P}_0, \mathcal{P}_1, \ldots, \mathcal{P}_k$ such that $\cup_{i=0}^{k} \mathcal{P}_i = \mathcal{P}$. An ordered $n$-tuple $(c_1, c_2, \ldots, c_n)$ is called a *chain*, with length $n-1$, of the poset $\mathcal{P}$ if $c_i \neq c_{i+1}$ and $c_i \leq c_{i+1}$ for $1 \leq i \leq n$. A chain $(c_1, c_2, \ldots, c_n)$ is *saturated* if it cannot be internally extended, that is $(c_i, c_{i+1})$ is an edge in the Hasse diagram of the poset for $1 \leq i \leq n$. A saturated chain is *symmetric* if it starts and ends at levels whose distance from the middle levels of the poset is the same.
- A poset has a *SSCD* if $\mathcal{P}$ can be decomposed into symmetric saturated chains.

- The above poset (lattice) appears in different guises in several branches of mathematics. For example, it is isomorphic to the poset of *Schubert cells* in the *Grassmannian* of $m$-planes in $\mathbb{C}^{m+n}$. In the groundbreaking paper [8], R. Stanley used the hard Lefschetz theorem to prove that $L(m, n)$ is rank-symmetric, unimodal, and strongly Sperner. Furthermore, he conjectured that it has a *SSCD*, that is, it can be expressed as a disjoint union of rank-symmetric, saturated chains. Despite many efforts the conjecture has been only proved for the cases where $min(m, n) \leq 4$.

The connection between Young's lattice and the lattice of threshold partitions of graphs associated with commutative rings needs to explored. Some progress has been made to write the lattice of threshold partitions as *SSCD*. However this progress is far away from our goal, which is to prove "Stanley's conjecture" for lattices of Young's type.

Thank you for your attention!

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