Apéry sets and the ideal class monoid of a numerical semigroup

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Background

Barucci and Khouja introduced the concept of ideal class semigroup associated to a numerical semigroup

They were mainly interested in the following aspects

- find bounds for the cardinality
- describe the generators
- study the reduction number

The ideal class monoid of a numerical semigroup

Let S be a numerical semigroup, that is, a submonoid of $(\mathbb{N}, +)$ such that $\mathrm{G}(S) = \mathbb{N} \setminus S$ has finitely many elements (gaps)

An **ideal** of S is a set I of integers such that

- $I + S \subseteq I$
- $z + I \subseteq S$ for some integer z

Let $\mathcal{I}(S)$ be the set of ideals of S

We write $I \sim J$ if there exists $z \in \mathbb{Z}$ such that $I = z + J$

The **ideal class monoid** of S is

$$\mathcal{C}\ell(S) = \mathcal{I}(S)/\sim$$

Addition is defined as $[I] + [J] = [I + J]$

First properties

Let

$$\mathcal{I}_0(S) = \{I \in \mathcal{I}(S) : \min(I) = 0\}$$

It follows easily that

$$\mathcal{C}\ell(S) \cong \mathcal{I}_0(S), [I] \mapsto -\min(I) + I$$

For $I \in \mathcal{I}_0(S)$, there exists $g_1, \ldots, g_k \in \mathrm{G}(S)$ such that

$$I = \{0, g_1, \ldots, g_k\} + S$$

Moreover, $\{g_1, \ldots, g_k\}$ can be taken to be an anti-chain with respect to

$$a \leq_S b \text{ if } b - a \in S$$

From this we can derive that

$$2^{\mathrm{m}(S)-1} + \mathrm{g}(S) - \mathrm{m}(S) + 1 \leq |\mathcal{C}\ell(S)| \leq 2^{\mathrm{g}(S)} - 2^{\mathrm{g}(S)-\mathrm{t}(S)} + 1$$

Apéry sets

Let S be a numerical semigroup with multiplicity m, and let $I \in \mathcal{I}_0(S)$

$$\mathrm{Ap}(I) = \{i \in I : i - m \notin I\}$$

Notice that if $i \in I$, then $i + km \in I$ for every non-negative integer k; thus

$$\mathrm{Ap}(I) = \{w_0(I) = 0, w_1(I), \ldots, w_{m-1}(I)\}$$

where $w_i(I) = \min(I \cap (i + m\mathbb{N}))$

Observe that $I = \mathrm{Ap}(I) + S$

$A = \{0, w_1, \ldots, w_{m-1}\} = \mathrm{Ap}(I)$ for some $I \in \mathcal{I}_0(S)$ if and only if $w_i + w_j(S) \geq w_{i+j}$ for all $i, j \in \{0, \ldots, m-1\}$ ($i + j$ taken modulo m)

Kunz coordinates

For every $i \in \{0, \ldots, m-1\}$, $w_i(I) = k_i(I)m + i$

The tuple $(k_1(I), \ldots, k_{m-1}(I))$ are the **Kunz coordinates** of I

A tuple $(x_1, \ldots, x_{m-1})$ are the Kunz coordinates of an ideal in $\mathcal{I}_0(S)$ if and only if

$$\begin{aligned}
&x_i \leq k_i(S), \text{ for all } i \in \{1, \ldots, m-1\}, \\
&x_{i+j} - x_i \leq k_j(S), \text{ for every } i, j \in \{1, \ldots, m-1\}, i+j < m, \\
&x_{i+j-m} - x_i \leq k_j(S) + 1, \text{ for every } i, j \in \{1, \ldots, m-1\}, i+j > m.
\end{aligned}$$

In particular,

$$|\mathcal{C}\ell(S)| \leq (k_1(S)+1) \times \cdots \times (k_{\mathrm{m}(S)-1}(S)+1)$$

Canonical ideal

Let S be a numerical semigroup. The **canonical ideal** of S is

$$\mathrm{K}(S) = \{x \in \mathbb{Z} : \mathrm{F}(S) - x \notin S\}$$

Let $f = \mathrm{F}(S) \mod \mathrm{m}(S)$

Then $I = \mathrm{K}(S)$ if and only if

$$w_i(I) = w_f(S) - w_j(S)$$

for all $i, j \in \{0, \dots, \mathrm{m}(S) - 1\}$ with $i + j \equiv f \pmod{\mathrm{m}(S)}$

In particular,

$$\mathrm{K}(S) = \mathrm{F}(S) - \mathrm{Maximals}_{\leq_S}(\mathbb{Z} \setminus S) + S$$

Reduction number

Let I be an ideal of a numerical semigroup S with multiplicity m

The **reduction number** of I, $\mathrm{r}(I)$, is the least non-negative integer r such that $(r+1)I = rI$

If g is a gap of S, then

$$\mathrm{r}(\{0,g\}+S) = \min\{k \in \mathbb{N} : (k+1)g \in S\}$$

If $\{a_1,\dots,a_h\} \subseteq \{1,\dots,m-1\}$, then

$$\mathrm{r}(\{0,a_1,\dots,a_h\}+S) \leq m-h$$

Hasse diagram of $(\mathcal{I}_0(S), \subseteq)$

Given $I, J \in \mathcal{I}_0(S)$, we have that $I \subseteq J$ if and only if $(k_1(J), \ldots, k_{m-1}(J)) \leq (k_1(I), \ldots, k_{m-1}(I))$

- $\min_{\subseteq}(\mathcal{I}_0(S)) = S$
- $\max_{\subseteq}(\mathcal{I}_0(S)) = \mathbb{N}$
- $|\operatorname{Maximals}_{\subseteq}(\mathcal{I}_0(S) \setminus \{\mathbb{N}\})| = \mathrm{m}(S) - 1$
- $|\operatorname{Minimals}_{\subseteq}(\mathcal{I}_0(S) \setminus \{S\})| = \mathrm{t}(S) = |\operatorname{Maximals}_{\leq_S}(\mathbb{Z} \setminus S)|$
- The length of the maximal strictly ascending chain is $\mathrm{g}(S) + 1 = |\mathbb{N} \setminus S| + 1$

Example

$S = \langle 4, 6, 9 \rangle$

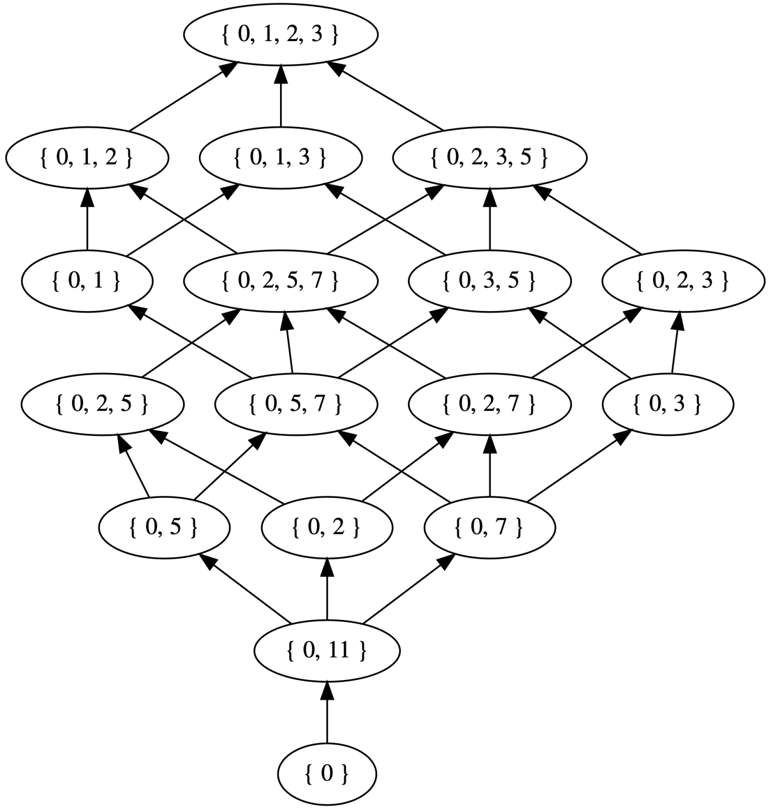
Maximal non-trivial ideals are of the form

$\{0, 1, \ldots, i-1, i+m, i+1, \ldots, m-1\} + S$

Minimal non-tivial ideals are

$\{0, f\} + S$ with $f \in \mathrm{Maximals}_{\leq_S}(\mathbb{Z} \setminus S)$

https://numerical-semigroups.github.io/



Irreducibles, atoms, quarks and primes

Let S be a numerical semigroup

The monoid $(\mathcal{I}_0(S), +)$ is reduced (the only unit is S), and it is highly non-cancellative

On $\mathcal{I}_0(S)$ we write $I \preceq J$ if there exists K such that $I + K = J$

An ideal $I \in \mathcal{I}_0(S)$, $I \neq S$, is (using Tringali's terminology)

- **irreducible** if $I \neq J + K$ for all non-units J and K such that $J \prec I$ and $K \prec I$
- an **atom** if $I \neq J + K$ for all non-units J and K
- a **quark** if there is no non-unit J with $J \prec I$
- a **prime** if $I \preceq J + K$ for some J, K implies that $I \preceq J$ or $I \preceq K$

Irreducibles are generators

An ideal I is irreducible if and only if $I \neq J + K$ for any non-units J and K with $J \neq I \neq K$

Every ideal in $\mathcal{I}_0(S)$ can be expressed as a sum of irreducible ideals

Example

For $S = \langle 5, 6, 8, 9 \rangle = \mathbb{N} \setminus \{1, 2, 3, 4, 7\}$

- Irreducibles: $\{0, g\} + S$ with g a gap, $\{0, 1, 3\} + S$, and $\{0, 3, 4\} + S$
- Atoms: $\{0, 3, 4\} + S$
- Quarks: $\{0, 3, 4\} + S$, $\{0, 3\} + S$, $\{0, 4\} + S$, $\{0, 7\} + S$
- No primes

Quarks

Let S be a numerical semigroup

Quarks are either

- minimal ideals with respecto to inclusion: $\{0, g\} + S$ with $g \in \mathrm{Maximals}_{\leq_S}(\mathbb{Z} \setminus S)$
- irreducible non minimal ideals such that for every $g \in \mathrm{Maximals}_{\leq_S}(\mathbb{Z} \setminus S)$, $g + I \subsetneq I$

Idempotent quarks correspond to unitary extensions of S

A numerical semigroup S is irreducible (symmetric or pseudo-symmetric) if and only if $\mathcal{I}_0(S)$ has at most two quarks

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Thank you for your attention