Null Ideals of Subsets of Matrix Rings

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Basic Problem

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Let $A_1, A_2, \ldots, A_k$ be $n \times n$ matrices with entries from F.
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  Take $f = \mu_1 \mu_2 \cdots \mu_k$.
- We really just need the least common multiple of all the min. polys.
  Let $\phi = \text{lcm}(\mu_1, \ldots, \mu_k)$.
  - ϕ is the unique monic polynomial in $F[x]$ of minimal degree that kills all the A_i
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All the polynomials above have coefficients from F.
What if we allow polynomials with matrix coefficients?

Null Ideals

Let R be a ring (associative, with identity, not necessarily commutative) and $S \subseteq R$.

Definition

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- So, $h(a) = cda^2$ while $f(a)g(a) = cada$
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Problem V.2: Understand null ideals of subsets of matrix rings.

Basic Properties of Null Ideals

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1. For any ring R, $N_R(S)$ is a left $R[x]$-module.
2. When R is commutative, then $N_R(S)$ is a (two-sided) ideal of $R[x]$.
3. When R is an integral domain, $N_R(S) \neq \{0\}$ if and only if S is finite.
4. When R is an integral domain and $S = \{a_1, \ldots, a_k\}$, $N_R(S)$ is generated by $(x - a_1) \cdots (x - a_k)$.

Each of #2, #3, and #4 can fail if R is noncommutative.

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Example: #4 need not hold if R is noncommutative.

- Let $a, b \in R$ be such that $ab \neq ba$.
- Let $h(x) = (x - a)(x - b) = x^2 - (a + b)x + ab$.

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- So, $(x - a)(x - b) \notin N_R(\{a, b\})$

Matrix Examples

Let F be a field, $\text{char}(F) \neq 2$. Let $R = M_2(F)$, the ring of 2×2 matrices over F

Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$, and $C = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

Let $S = \{A, B\}$ and $T = \{A, C\}$

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  - But, $A - C$ is invertible $\rightsquigarrow \alpha = 0 \rightsquigarrow \beta = 0$
- In fact, $N(T)$ is generated (as a two-sided ideal of $R[x]$) by x^2

Connection to Integer-valued Polynomials

Let D be a (commutative) integral domain with field of fractions K. Let $S \subseteq D$. Then,

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$$\begin{aligned} f \in \text{Int}(S, D) &\iff \frac{g(s)}{d} \in D \text{ for all } s \in S \\ &\iff g(s) \in dD \text{ for all } s \in S \\ &\iff \overline{g} \in N_{\overline{D}}(\overline{S}) \end{aligned}$$

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Null ideals can give information about $\text{Int}(S, D)$ **even in noncommutative settings**.

Integer-valued Polynomials over Matrix Rings

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We will make a "matrix version" of $\text{Int}(S, D)$.

- $M_n(D)$: $n \times n$ matrices over D, $M_n(K)$: $n \times n$ matrices over K
- Embed $K \hookrightarrow M_n(K)$ by $a \mapsto$ scalar matrix aI

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- $\text{Int}(S, M_n(D))$ is closed under addition (easy)

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- Is it closed under multiplication?

Integer-valued Polynomials over Matrix Rings

D: integral domain, K: fraction field of D, $S \subseteq D$

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Back to Null Ideals

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Theorem

$\mathrm{Int}(S, M_n(D))$ is a ring if and only if $N_{M_n(\overline{D})}(\overline{S})$ is a two-sided ideal of $M_n(D/dD)[x]$ for each $d \neq 0$.

Focus on Matrices

For the remainder of the talk, we will assume:

- F is a field, $R = M_n(F)$, and $S \subseteq M_n(F)$
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We will focus on Question #1.

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Definitions

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Problem V.3: Classify/characterize the core subsets of $M_n(F)$.

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2. Assume that S is a full conjugacy class.
   That is, $S = \{UAU^{-1} \mid U \in GL(n, F)\}$ for some $A \in M_n(F)$.
   Then, S is core.

Example: Intersections of Core Sets Need Not be Core

Let F be a field, $\text{char}(F) \neq 2$

Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$, and $C = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

Let $S_1 = \{A, C\}$ and $S_2 = \{B, C\}$

- $\phi_{S_1}(x) = x^2$ and $\phi_{S_2}(x) = x^2$
- Neither $N(S_1)$ nor $N(S_2)$ contains a linear polynomial.
  (Ultimately, this is because both $A - C$ and $B - C$ are invertible.)
  Thus, both S_1 and S_2 are core (both are generated by x^2)
- However, $S_1 \cap S_2 = \{C\}$, which is not core.

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- In $M_2(F)$, "conjugacy class" = "minimal polynomial class"
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Overall (and ultimately successful!) strategy to decide if S is core:

1. Partition S into conjugacy classes:

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3. Figure out what happens when the S_i are combined back into the original S.
   This gets wild.

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3. Assume F is a finite field with q elements. If $|S| \geq q + 1$, then S is core.

Combining Classes back into S

$$S = S_1 \cup S_2 \cup \cdots \cup S_k$$
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3. **Distinct root case**: Assume $m_i(x) = (x - a)(x - b)$ for $a, b \in F$ with $a \neq b$.
   Then, **S may or may not be core**.
   It depends on the other classes S_j with $j \neq i$.
   (This is the "wild" case.)

Some Confounding Examples

Assume $\text{char}(F) \neq 2$ and let

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$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x(x+1) \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x(x-1)$$

  So, T is not core
- Why is S core but T is not core????

Some Confounding Examples

$$A_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \; A_2 = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad A_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; A_4 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$S = \{A_1, A_2, A_3\}, \qquad T = \{A_1, A_2, A_4\}$$

$$\phi_S(x) = x(x-1)(x+1) \qquad \phi_T(x) = x(x-1)(x+1)$$

Why is S core but T is not core????

Sketch of an answer:

- We need to look at left annihilators of translations $A - a$, where a solves μ_A

	Translate by 0	Translate by 1	Translate by -1
A_1	$A_1 - 0 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$	$A_1 - 1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$	
A_2	$A_2 - 0 = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$		$A_2 + 1 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$
A_3		$A_3 - 1 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & -2 \end{smallmatrix}\right)$	$A_3 + 1 = \left(\begin{smallmatrix} -2 & 0 \\ 0 & 0 \end{smallmatrix}\right)$
A_4		$A_4 - 1 = \left(\begin{smallmatrix} -2 & 0 \\ 0 & 0 \end{smallmatrix}\right)$	$A_4 + 1 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 2 \end{smallmatrix}\right)$

- The matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ is in the left annihilator of each of $A_1 - 0$, $A_2 + 1$, and $A_4 + 1$. So, $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) x(x+1) \in N(T)$
- To obtain a similar element in $N(S)$, we need to translate by 0, 1, and -1. The resulting polynomial is a multiple of $x(x-1)(x+1)$.

Algorithm to decide if a finite subset of $M_2(F)$ is core

Given a finite set $S \subseteq M_2(F)$:

1. Partition S into conjugacy classes $S = S_1 \cup \cdots \cup S_k$.
   For each i, let $\phi_i = \phi_{S_i}$. Then, $\deg \phi_i \leq 2$.

2. Determine whether each S_i is core.
   - If each S_i is core, then S is core.
   - If some S_i is not core and ϕ_i is either irreducible quadratic or quadratic with a repeated root, then S is not core.

3. Let S_0 be the union of all the S_i that are core.

   Let $T = S \setminus S_0$. Then, T is a union of non-core classes, and each class corresponds to a min. poly. of the form $(x-a)(x-b)$ with $a \neq b$.

   Examine the left annihilators of translates of elements of T.

   These annihilators can allow us to determine whether S is core.

Is there a better method to identify core sets?

Summary

- There is a connection between null ideals and integer-valued polynomials. This holds even in noncommutative settings! (e.g. for matrix rings)

- Solved problem: Determine all the finite core subsets of $M_2(F)$

Open problems:

1. For an integral domain D, which subsets $S \subseteq M_n(D)$ are such that $\text{Int}(S, M_n(D))$ is a ring?
   - Are null ideals the best method to find these subsets?

2. Enumerate or estimate the number of core subsets.
   - Are core subsets common? Are they sparse?
   - When F is finite, how many core subsets does $M_2(F)$ contain?

3. Classify/describe the infinite core subsets of $M_2(F)$.

4. Identify generators of non-core subsets of $M_2(F)$.

5. Explore null ideals and core subsets of $M_n(F)$ for $n \geq 3$.

THANK YOU!!

References

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