# On the Zaks Property of Polynomial Extensions

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In this talk, $R$ is an integral domain with fraction field $K \neq R$. We assume that $R$ is atomic, that is, each nonzero nonunit in $R$ is a finite product of atoms (irreducible elements).

# On general Zaks domains

The domain $R$ is a **Zaks** domain if there exists a factorial domain $D$ containing $R$ such that each irreducible element of $R$ remains irreducible in $D$.
A Zaks domain is half-factorial, but the converse is not true in general. Recall that the domain $R$ is **half-factorial** if all factorizations of a nonzero element of $R$ into irreducible elements have the same length.

# Reduction to overrings

$R$ is a Zaks domain if and only if $R$ has a factorial overring $D$ such that $R \underset{\mathrm{Irr}}{\subseteq} D$ (that is, each irreducible element of $R$ remains irreducible in $D$).

Hence a polynomial extension $R[\mathbf{X}]$ is a Zaks domain if and only there exists a factorial overring $D$ of $R$ and a factorial overring $T$ of $D[\mathbf{X}]$ such that $R[\mathbf{X}] \underset{\mathrm{Irr}}{\subseteq} T$. Here $\mathbf{X}$ is a set of independent indeterminates, not necessarily finite.

**Remark 1.**

*If $R$ is not factorial, then $T \neq D[\mathbf{X}]$.*

Indeed, $R$ is not factorial if and only if there exist two atoms $a, b$ in $R$ that are associated in $D$, but not in $R$. Hence, the polynomial $aX + b$ is irreducible in $R[X]$, but not in $D[X]$. □

Actually, this example shows that $T[X]$ is not a well-centered extension of $R[X]$, which is a necessary condition for having $R[X] \underset{\mathrm{Irr}}{\subseteq} D[X]$. Recall that a domain $D$ containing $R$ is a **well-centered** extension of $R$ if $D = \mathrm{U}(D)R$.

# A characterization of Zaks domains

Recall that an element $c$ of $R$ is **reducible** if $c$ is a product of two nonunits in $R$.

**Theorem 2.**

*Let $D$ be a factorial overring of $R$. The following three conditions are equivalent:*

1. $R \underset{\mathrm{Irr}}{\subseteq} D$.
2. *The following three conditions are satisfied:*
   - 2.1 *$D$ is a well-centered extension of $R$.*
   - 2.2 *$R \underset{\mathrm{U}}{\subseteq} D$ (that is, each non-invertible element of $R$ is non-invertible also in $D$).*
   - 2.3 *If $c$ is reducible in $R$, and $u \in D, uc \in R$, then $uc$ is reducible in $R$.*
3. *$R$ is half-factorial, and for each $t \in D^\bullet$, we have $\ell_R(t) = 0 \Leftrightarrow t \in \mathrm{U}(D)$.*

*Hence $R$ is Zaks domain if and only if there exists an overring $D$ of $R$ satisfying the above conditions.*

# Remark

Theorem 2.3 shows what additional property one can add to half-factoriality in order to obtain the Zaks property. There are also other possibilities.

# A characterization of Zaks polynomial rings

Using the characterization of Zaks domains, we obtain:

**Theorem 3.**

*Let $R[\mathbf{X}]$ be a proper polynomial extension of a domain $R$ with field of fractions $K$, and let $D$ be a factorial overring of $R$ such that $R \underset{\mathrm{Irr}}{\subseteq} D$. Set $W = \{g \in \mathrm{Irr}(D[\mathbf{X}]) \mid (\forall u \in \mathrm{U}(D))\ ug \notin R[\mathbf{X}]\}$. Let $S$ be the multiplicative subset of $D[\mathbf{X}]$ generated by the set $W$, and let $\mathrm{Sat}_{D[X]}(S)$ be the saturation of $S$ in $R$. The following three conditions are equivalent:*

1. $R[\mathbf{X}] \underset{\mathrm{Irr}}{\subseteq} D[\mathbf{X}]_S$.
2. 2.1 $R \underset{\mathrm{Irr}}{\subseteq} D$.

   2.2 *If $f$ is a reducible polynomial of positive degree in $R[\mathbf{X}]$, and $\varphi \in S$, then $f\varphi$ is reducible in $R[\mathbf{X}]$.*
3. $R[\mathbf{X}]$ *is half-factorial, and (the saturation of $S$),* $\mathrm{Sat}_{D[X]}(S) = \{f \in R[\mathbf{X}] \mid \ell_{R[\mathbf{X}]}(f) = 0\}$.

*Hence, $R[\mathbf{X}]$ is a Zaks domain if and only if there exists a factorial domain $D$ as above.*

# Remark

Here is a possible reformulation of Theorem 3.2:
The following three conditions are satisfied:

1. $D$ is a well-centered extension of $R$.
2. $R \underset{\mathrm{U}}{\subseteq} D$ (that is, each non-invertible element of $R$ is non-invertible also in $D$).
3. If $f$ is a reducible polynomial (not necessarily of positive degree) in $R[\mathbf{X}]$, and $\varphi \in S$, then $f\varphi$ is reducible in $R[\mathbf{X}]$.

# Zaks not-factorial polynomial rings

We show that all polynomial extensions of a local one-dimensional Mori domain obtained by Valentina Barucci in a more general setting, are Mori and Zaks, but not factorial. However, Zaks proper polynomial extensions of Krull domains are factorial. It is not clear if this is true more generally for completely integrally closed domains. Recall that a domain is Mori if it satisfies the ascending chain condition on integral divisorial ideals. A domain is Krull if and only is both completely integrally closed and Mori.

**Lemma 4.**

*Let $A \subseteq B$ be domains such that $A$ is integrally closed in $B$. Then all factors in $B[\mathbf{X}]$ of a polynomial $f \in A[\mathbf{X}]$ of positive degree belong to $A[\mathbf{X}]$ in each of the following two cases:*

1. *$f$ is a monic polynomial in $A[X]$ (so $f$ is a polynomial in one indeterminate).*
2. *$A, B$ are fields.*

Proof.

1. Let $g$ be a monic divisor of $f$ in $R[X]$ of positive degree. Since $f$ is a monic polynomial in $R[X]$ and $R$ is integrally closed, all roots of $f$, so also of $g$ in a splitting field of $f$ over $B$, belong to $A$. Hence also the coefficients of $g$ belong to $A$, so $g \in A[X]$. Since $f$ is irreducible in $R[X]$, it follows that $f = g$, so condition (2) holds.
2. We may assume that $\mathbf{X}$ is finite: $\mathbf{X} = \{X_1, \ldots, X_n\}$. We proceed by induction on $n$. If $n > 1$, then, by changing indeterminates, we may assume that $f$ is a polynomial in $A[X_1, \ldots, X_{n-1}][X_n]$ with invertible leading coefficient, so we may apply item (1).

$\square$

## Proposition 5.

*Let* $R \subseteq D$ *be local domains with the same maximal ideal* $M$*, where* $D$ *is factorial. Let* $D[\mathbf{X}]$ *be a proper polynomial extension of* $D$*. Set*

$$S = \{g \in (D[\mathbf{X}] \mid (\forall u \in \mathrm{U}(D))\; ug \notin R[\mathbf{X}]\}.$$

*The following two conditions are equivalent:*

1. $S$ *is a multiplicatively closed subset of* $R[\mathbf{X}]$*, and* $R[\mathbf{X}] \underset{\mathrm{Irr}}{\subseteq} D[\mathbf{X}]_S$.
2. $R$ *is integrally integrally closed (equivalently, the field* $R/M$ *is algebraically closed in* $D/M$*).*

*Hence all polynomial extensions of* $R$ *are Zaks domains, equivalently half-factorial, if and only if* $R$ *is integrally closed.*

# Proof: (1) $\Rightarrow$ (2)

Since $R[\mathbf{X}] \underset{\mathrm{Irr}}{\subseteq} D[\mathbf{X}]_S$, we see that $R[\mathbf{X}]$ is half-factorial. By Coykendall's Theorem, we obtain that $R$ is integrally closed. Since $D$ is an overring of $R$, we see that $R$ is integrally closed in $D$, and this holds if and only if the field $R/M$ is algebraically closed in $D/M$.

## Proof: (2) $\Rightarrow$ (1)

Let $g_1, g_2 \in S$, so $g_1, g_2 \notin R[\mathbf{X}]$. Assume that $g_1g_2 \notin S$, so $ug_1g_2 \in R[\mathbf{X}]$ for some $u \in \mathrm{U}(D)$. Since $ug_1 \in S$, replacing $g_1$ by $ug_1$, we may assume that $g_1g_2 \in R[\mathbf{X}]$. Since $g_1 \notin R[\mathbf{X}]$, we obtain that $g_1 \notin M$. Also, $g_1 \notin \mathrm{U}(D) = D \setminus M$, so $g_1 \notin D$. Applying Lemma 2 to the fields $(R/M)[\mathbf{X}] \subseteq (D/M)[\mathbf{X}]$, we obtain that $g_1 \in R[\mathbf{X}]$, a contradiction. Hence $S$ is a multiplicative subset of $D[\mathbf{X}]$.

Let $f$ be a reducible polynomial in $R[\mathbf{X}]$, thus $f = f_1f_2$, where $f_1, f_2$ are noninvertible polynomials in $R[\mathbf{X}]$. Let $\varphi \in S$. We have to show that if $f_1f_2\varphi \in R[\mathbf{X}]$, then $f_1f_2\varphi$ is reducible in $R[\mathbf{X}]$. If, e.g., $f_2 \in M[R][\mathbf{X}]$, then $f = f_1(f_2\varphi)$ is reducible in $R[\mathbf{X}]$. We now assume that $f_1, f_2 \notin MR[X]$. Hence $f_1, f_2$ have unit content in $R$, so the content of $\varphi$ over $R$ (the submodule of $D$ over $R$ generated by the coefficients of $\varphi$) is contained in $R$, implying that $\varphi \in R[\mathbf{X}]$, a contradiction. $\square$

# Example

*Let $F \subsetneq L$ be fields such that $F$ is uncountable and algebraically closed in $L$. Let $t$ be an indeterminate over $L$. Set $D = L[t]_{(t)}, M = L[t]_{(t)}$, and $R = F + M$. Then all polynomial extensions of $R$ are Mori and Zaks.*

Indeed, by Proposition 5, all polynomial extensions of $R$ are Zaks. It is known that $R = F + M$ is Mori, and since the field $F$ is uncountable, it follows that all polynomial extensions of $R$ are Mori. It is possible that the assumption that $F$ is uncountable, is superfluous.

# The property of $R[X]$ being integrally closed in terms of half-factoriality, for $R$ half-factorial

We present a simple proof to an extension of *Coykendall's Theorem that a half-factorial domain is integrally closed*. Recall that a polynomial in $R[X]$ is *primitive* if its content over $R$ is not contained in a proper principal ideal of $R[X]$.
The next theorem is well-known.

**Theorem 6.**

*Let $R$ be a half-factorial domain with fraction field $K$. The following conditions are equivalent:*

1. *$R$ is integrally closed.*
2. *Each irreducible monic polynomial $R[X]$ of positive degree is irreducible also in $K[X]$.*
3. *Each irreducible monic polynomial in $R[X]$ of positive degree is prime in $R[X]$.*
4. *Each monic polynomial in $R[X]$ of a positive degree is a product of prime polynomials.*

# Proof

(1) $\Rightarrow$ (2) This follows from Lemma 4.1.
(2) $\Rightarrow$ (1) Let $c \in \overline{R}$ (the integral closure of $R$). The minimal polynomial $g$ of $c$ over $R$ is irreducible in $R[X]$, so also in $K[X]$. Thus $g = X - c$, so $c \in R$, and $R$ is integrally.
(2) $\Rightarrow$ (3) Assume that $f \in \mathrm{Irr}(R[X])$ is monic of positive degree. Let $g, h$ be polynomials in $R[X]$ such that $f \mid gh$ in $R[X]$. Since $f$ is irreducible in $K[X]$, we see that $f$ is prime in $K[X]$, so we may assume that $f \mid g$ in $K[X]$. Since $f$ is monic, when dividing $f$ by $g$ with remainder in $R[X]$ and in $K[X]$, we obtain the same remainder. It follows that $f \mid g$ in $R[X]$, so $f$ is prime in $R[X]$.
(3) $\Rightarrow$ (2) Clear.
(3) $\Leftrightarrow$ (4) Clear.

**Theorem 7.**
*Let $R$ be a half-factorial domain. The following conditions are equivalent:*

1. *$R$ is integrally closed.*
2. *The multiplicative monoid consisting of monic polynomials, is half-factorial (that is, all factorizations into atoms of a monic polynomial in $R[X]$ have the same length).*

# Proof

(1) $\Rightarrow$ (2) By Theorem 6 [(1) $\Rightarrow$ (4)], each monic polynomial in $f \in R[X]$ of positive degree is a product of prime polynomials, so any two factorizations of $f$ are identical (up to order).

(2) $\Rightarrow$ (1) Assume that $R$ is not integrally closed. By Theorem 6, there exists a polynomial $f$ of minimal degree among the monic polynomials of positive degree that are irreducible in $R[X]$, but not in $K[X]$. Let $f = g_1 g_2 \dots g_n$, where $n > 1$, and $g_i$ are irreducible monic polynomials in $K[X]$. For each $i$, choose $a_i \in R^\bullet$, such that $a_i g_i \in R[X]$ with $\ell_R(a_i)$ minimal. For all $i$, the polynomial $a_i g_i$ is primitive in $R[X]$. Since $a_i g_i$ is irreducible in $K[X]$, it follows that $a_i g_i$ is irreducible in $R[X]$. If for some $i$, we have $a_i \in \mathrm{U}(R)$, then $g_i \in R[X]$, and $\frac{f}{g_i} \in R[X]$. If $n > 2$, then $\frac{f}{g_i}$ is reducible in $R[X]$, contradicting the minimality of $\deg f$. If $n = 2$ we get the contradiction that $f = g_i \frac{f}{g_i}$ is reducible in $R[X]$. It follows that all $a_i$ are non-units in $R$, so $(\prod_{i=1}^n a_i) f$ has a factorization of length $\geq n + 1$. On the other hand $(\prod_{i=1}^n a_i) f$ is equal to $\prod_{i=1}^n a_i g_i$, which is a factorization of length $n$.

# Remark

In Theorem 6 and 7, we may replace monic polynomials by polynomials with invertible initial coefficients, where the initial coefficient of a polynomial $f$ is $f(0)$. Indeed, we use reciprocity. Recall that the reciprocal to a polynomial $a_n X^n + \cdots + a_0$, where $a_0$ and $a_n$ are nonzero, is the polynomial $a_0 X^n + \cdots + a_n$. For example, we may conclude from Theorem 7 for $R$ half-factorial, that $R$ is integrally closed $\Leftrightarrow$ the multiplicative monoid consisting of polynomials with invertible initial coefficients, is half-factorial .