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On the Apéry algorithm for a plane singularity

Graz - Conference on Rings and Factorizations

July 2023

0. Algebroid branches and curves

An algebroid branch is a one-dimensional domain of the form

$$R = k[[x_1, \ldots, x_n]]/P$$

(P prime ideal, k algebraically closed).

$Q(R) \cong k((t))$ and $\overline{R} \cong k[[t]]$ (and it is a finite R-module).

If v is the usual valuation on $k((t))$, then $v(R \setminus \{0\})$ is a numerical semigroup. i.e. a submonoid $S \subseteq (\mathbb{N}, +)$ s.t.

$$|\mathbb{N} \setminus S| < \infty \ .$$

$S = \langle g_1, \ldots, g_\nu \rangle = \{\sum_i n_i g_i : \ n_i \in \mathbb{N}\}$, where $\text{GCD}(g_1, \ldots, g_\nu) = 1$.

An algebroid curve is a one-dimensional, reduced ring of the form
$R = k[[x_1, \ldots, x_n]]/P_1 \cap \cdots \cap P_h$
(P_i height $n-1$ primes, k algebraically closed).

$R_i = k[[x_1, \ldots, x_n]]/P_i$ is the i-th algebroid branch of R.

$Q(R) \cong k((t_1)) \times \cdots \times k((t_h))$ and $\overline{R} \cong k[[t_1]] \times \cdots \times k[[t_h]]$.
Remark. $k[[t_i]] = \overline{(R/P_i)}$.

If we set $v(r) = (v_1(r_1), \ldots, v_h(r_h))$, with v_i the usual valuation on $k((t_i))$, then the value semigroup is:

$$S = v(R) := \{v(r) : \ r \in R, \ r \text{ non-zero divisor}\} \subset \mathbb{N}^h.$$

Remark. For plane curves we have rings of the form $k[[X, Y]]/(F)$. If F is irreducible we have a branch.

1. Value semigroups and equisingularity of plane curves.

Value semigroup is a possible criterion of equisingularity for algebroid branches or curves.

Two plane algebroid branches are formally equivalent (i.e. they have the same multiplicity sequence) $\Leftrightarrow$ they have the same value semigroup.
In case $k = \mathbb{C}$ two plane analytic branches are topologically equivalent $\Leftrightarrow$ are formally equivalent [Zariski].

Notice that any algebroid (resp analytic) plane branch is formally (resp. topologically) equivalent to an algebraic branch (i.e. F is a polynomial) [Samuel]

Multiplicity sequences and value semigroups of plane algebroid branches have been charachterized [Zariski, Bertin-Carbonne, Brezinsky, Angermüller].

As in the one branch case,
two plane algebroid curves are formally equivalent $\Leftrightarrow$
they have the same value semigroup [Waldi].

Is it possible to characterize value semigroups of plane curves?

Remark. Any numerical semigroup is the value semigroup of a branch (e.g. monomial curves). But there is no characterization of value semigroups of algebroid curves.

Remark. For non-plane singularities the different criteria are no more equivalent.

2. Why to study value semigroups? One branch case

Notation: $\mathfrak{m}$ max ideal of R, $S = v(R)$, $M = S \setminus \{0\}$,
$f(S)$ =max$(\mathbb{N} \setminus S)$ (Frobenius nb.), $n(S) = |\{s \in S \mid s < f(S)\}|$,

Proposition. If $I \supseteq J$ are two fractional ideals, then $\lambda_R(I/J) = |v(I) \setminus v(J)|$.

Using this fact we can read numerically many invariants and properties of the ring:

• degree of singularity: $\lambda_R(\bar{R}/R) = f(S) + 1 - n(S)$
(= number of holes)

• multiplicity: $e(R) = \lambda_R(R/(x))$ =min$M$
(x minimal reduction of $\mathfrak{m} \Leftrightarrow v(x)$ =min$M$)

Also we can get information on embeddig dimension, type, Goresteinness, Arf property, C.I. property, tangent cone etc.

3. Value semigroups of algebroid curves

The value semigroup of an algebroid curve is a submonoid of $\mathbb{N}^h$, with some more properties connected to valuations.

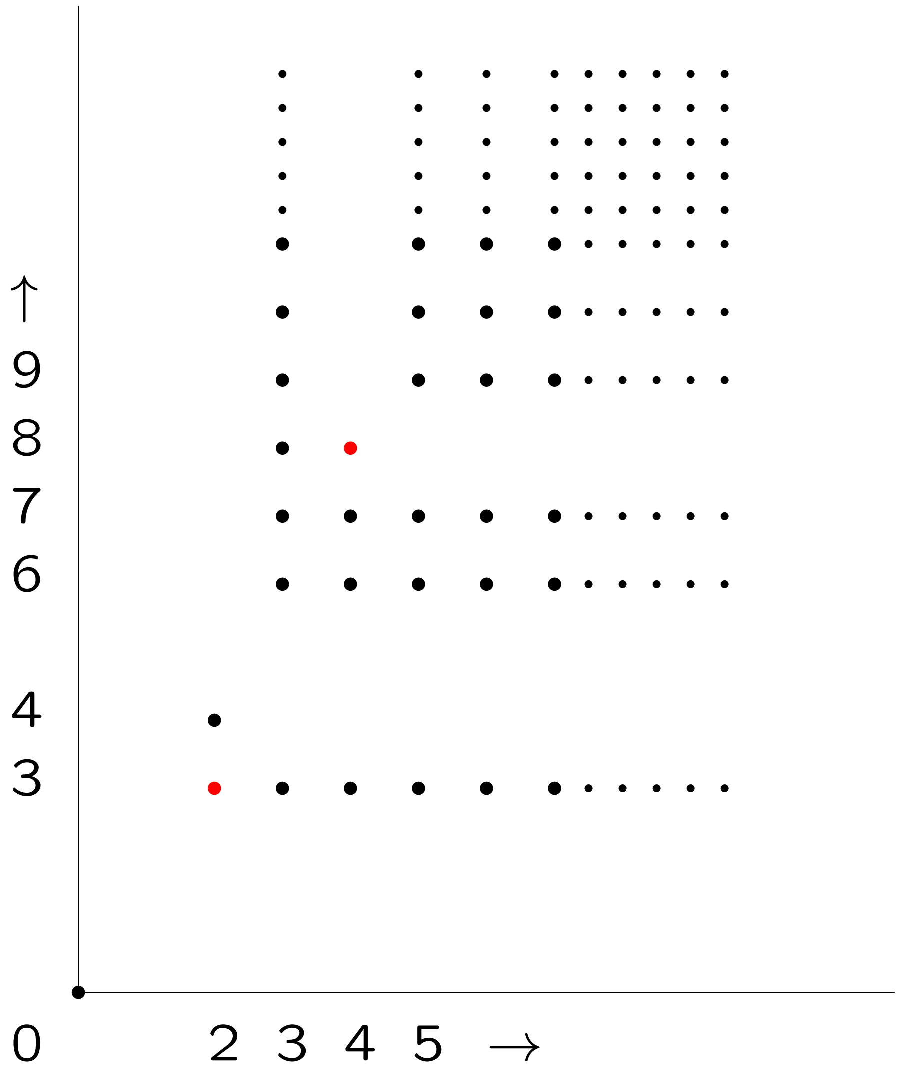
In the case $h = 2$, setting
$\Delta^S(a_1, a_2) = (\{(a_1, y) : a_2 < y\} \cup \{(x, a_2) : a_1 < x\}) \cap S$, they are:

(1) $\exists\ \boldsymbol{\gamma} = \boldsymbol{\gamma}(S) \in \mathbb{N}^2$ s.t. $\boldsymbol{\Delta}^S(\boldsymbol{\gamma}) = \emptyset$ and $\boldsymbol{\gamma} + (1, 1) + \mathbb{N}^2 \subseteq S$;

(2) $\boldsymbol{\alpha}, \boldsymbol{\beta} \in S \ \Rightarrow$ min$(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in S$;

(3)

(4) $(0, 0)$ is the only element of S on the axes.



Picture 1. $S = v(R)$

$$R = \frac{k[[x,y,z]]}{(x^3 - z^2, y) \cap (x^3 - y^4, z)}$$

$$x \mapsto (t^2, u^4)$$

$$y \mapsto (0, u^3)$$

$$z \mapsto (t^3, 0)$$

$$v(x+y) = (2,3)$$

$$\gamma = (4,8)$$

Formal definition with more branches: $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_h)$

$\Delta_i(\boldsymbol{\alpha}) = \{\boldsymbol{\beta} \mid \beta_i = \alpha_i, \beta_j > \alpha_j, \forall\ j \neq i\}$ $\qquad$ $\Delta_i^S(\boldsymbol{\alpha}) = \Delta_i(\boldsymbol{\alpha}) \cap S$

$\Delta(\boldsymbol{\alpha}) = \bigcup_i \Delta_i(\boldsymbol{\alpha})$ $\qquad$ $\Delta^S(\boldsymbol{\alpha}) = \Delta^S(\boldsymbol{\alpha}) \cap S$

(1) $\exists\ \boldsymbol{\gamma} = \boldsymbol{\gamma}(S) \in \mathbb{N}^h$ s.t. $\Delta^S(\boldsymbol{\gamma}) = \emptyset$ and $\boldsymbol{\gamma} + (1, \dots, 1) + \mathbb{N}^h \subseteq S$;

(2) $\boldsymbol{\alpha}, \boldsymbol{\beta} \in S \ \Rightarrow$ min$(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in S$;

(3) $\boldsymbol{\alpha} \neq \boldsymbol{\beta} \in S$ and $\alpha_i = \beta_i$ (for some i) $\Rightarrow$
$\exists\ \boldsymbol{\delta} \in S$ s.t. $\delta_i > \alpha_i = \beta_i$ and $\delta_j \geq$ min$\{\alpha_j, \beta_j\}$
(and the equality holds if $\alpha_j \neq \beta_j$).

(4) $(0, \dots, 0)$ is the only element of S with a zero component

4. Good semigroups

A subsemigroup S of $\mathbb{N}^h$ satisfying properties (1), (2), (3) is called a good semigroup. If (4) holds it is said local.

Remark. Properties (1), (2) and (3) imply that a good semigroup is completeley determined by its elements in the hyperrectangle bounded by $(0, \dots, 0)$ and $\gamma + (1, \dots, 1)$

Not all good semigroups arise as value semigroups [V. Barucci, _ , R. Fröberg - 2000], [N. Maugeri, G. Zito - 2019]

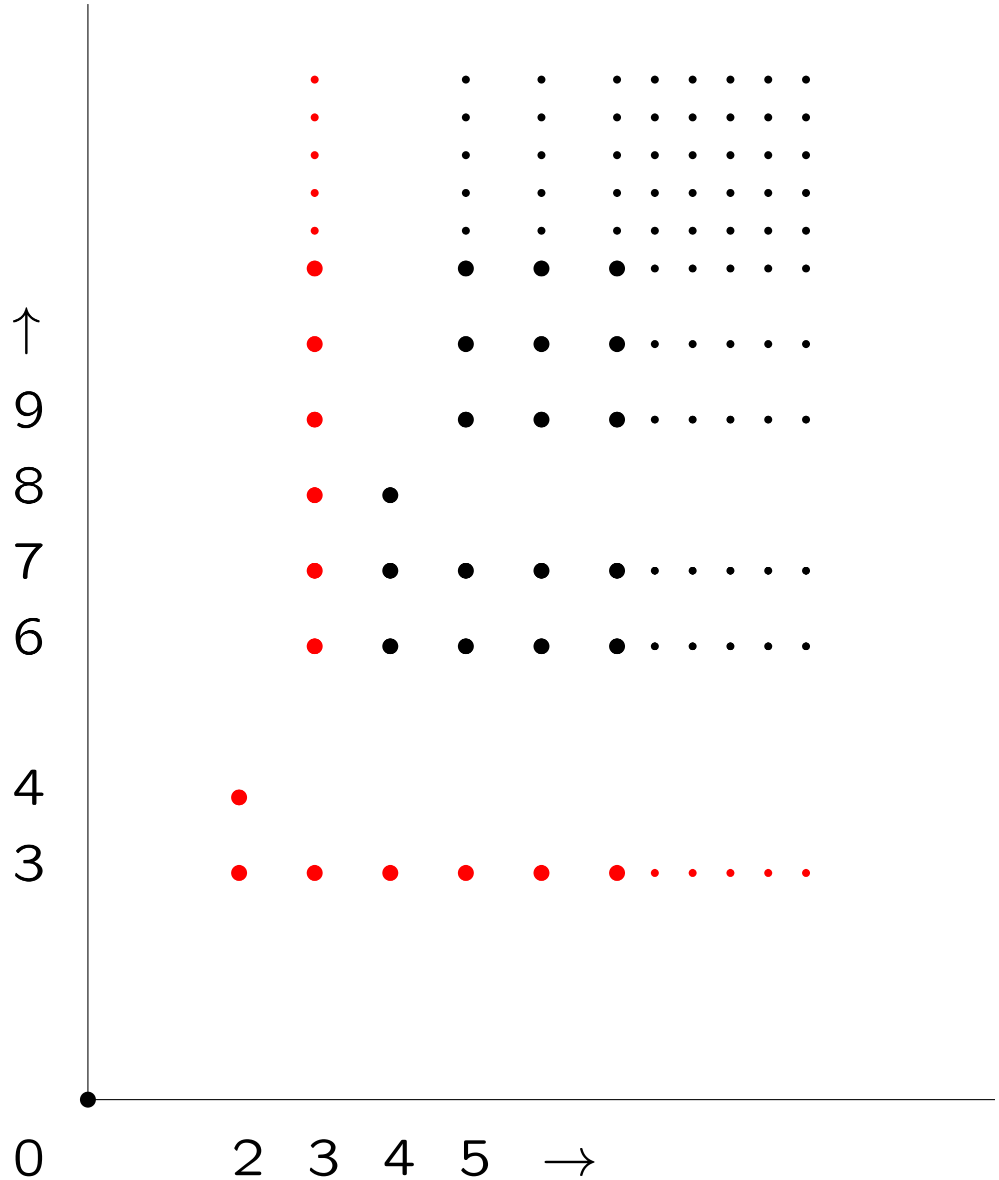
Open problem: characterize value semigroups among good semigroups.

If we want to define concepts or to prove results for good semigroups we cannot make use of valuation, so we have to use only numerical/combinatorical techniques.

Definition. $I \subseteq \mathbb{Z}^h$ is a relative ideal of S if $\boldsymbol{\alpha} + I \subseteq I$, $\forall\ \boldsymbol{\alpha} \in S$ and $\exists\ \boldsymbol{\alpha} \in S$, s.t. $\boldsymbol{\alpha} + I \subseteq S$.
We say that I is good if it satisfies properties (2), (3) ((1) follows by the same property for good semigroups).

Notation/remarks: • $\boldsymbol{m}(E)$:=min$E$;
• If E, F are relative ideals, $E + F := \{\boldsymbol{\alpha} + \boldsymbol{\beta} \mid \boldsymbol{\alpha} \in E, \boldsymbol{\beta} \in F\}$.
$E - F := \{\boldsymbol{\alpha} \in \mathbb{Z} \mid \boldsymbol{\alpha} + F \subseteq E\}$;
• I fractional ideal of $R \Rightarrow v(I)$ good relative ideal of $v(R)$.

"Bad" facts:
• good semigroups are not finitely generated as semigroups;
• good ideals are not finitely generated as semigroup ideals;
• operations on good ideals do not produce good ideals:
• we have to deal with infinite sets (e.g. $M \setminus 2M$).
• It is much more difficult to prove results for $h \geq 3$, than for $h = 2$. However, I do not know results proved in the case $h = 2$ that are false for $h \geq 3$, but many of them have been proved only for $h = 2$.



Picture 2. Generators of S

5. Why to study value semigroups in the general case?

Proposition. [_] Let I be a good relative ideal of S. Let $\leq$ be the usual partial order on $\mathbb{N}^h$. Then, $\forall\ \boldsymbol{\alpha}, \boldsymbol{\beta} \in I$, any saturated chain

$$\boldsymbol{\alpha} = \boldsymbol{\alpha}_0 < \boldsymbol{\alpha}_1 < \cdots < \boldsymbol{\alpha}_{m-1} < \boldsymbol{\alpha}_m = \boldsymbol{\beta}$$

($\boldsymbol{\alpha}_i \in I$) has the same length m.

Using this fact it is possible to define a "distance" function, $d(E \setminus F)$, between good relative ideals $E \supseteq F$:

Proposition. [_]
i) $\forall\ G \subseteq F \subseteq E$: $\quad d(E \setminus G) = d(E \setminus F) + d(F \setminus G)$.
ii) $\forall\ F \subseteq E$: $\quad d(E \setminus F) = 0 \Leftrightarrow E = F$.

Proposition. [_] If $I \supseteq J$ are two fractional ideals of R, then $\lambda_R(I/J) = d(v(I) \setminus v(J))$.

Invariants and properties of rings we can read on semigroups.

Notation. $M = S \setminus \{\mathbf{0}\}$ ad $\boldsymbol{e} = (e_1, \ldots, e_h) = \min(M)$.

- multiplicity: $\lambda_R(R/(x)) = e_1 + \cdots + e_h$

with x minimal reduction of $\mathfrak{m}$ i.e. $v(x) = \boldsymbol{e}$.
Notice that e_i is the multiplicity of the i-th branch of R;

- degree of singularity: $\lambda_R(\overline{R}/R) = d(\mathbb{N}^h \setminus S)$;

Also we can get information e.g on Goresteinness [Delgado], Arf property, embedding dimension [Maugeri, Zito], type [_, Guerrieri, Micale].

The study of other properties is still open:

- Properties of the tangent cone $\mathrm{gr}_{\mathfrak{m}}(R) = \bigoplus \mathfrak{m}^i/\mathfrak{m}^{i+1}$;
- complete intersection property;
- characterization of value semigroups of plane algebroid curves.

6. Blowing up tree and multiplicity tree

Let R be a branch: its blow up (or strict quadratic transform) is

$$R^{\mathfrak{m}} = \cup_{n>0}(\mathfrak{m}^n : \mathfrak{m}^n).$$

We have $\mathfrak{m}^n : \mathfrak{m}^n \subseteq \mathfrak{m}^{n+1} : \mathfrak{m}^{n+1}$ $(\forall n)$ and $R^{\mathfrak{m}} = \mathfrak{m}^{n_0} : \mathfrak{m}^{n_0}$ for some n_0, since R is Noetherian. Moreover, if x is a minimal reduction of $\mathfrak{m}$ and $\mathfrak{m} = (x, x_2, \ldots, x_\nu)$, $R^{\mathfrak{m}} = R[x_2/x, \ldots x_\nu/x]$.

It holds $R \subset R^{\mathfrak{m}} \subseteq \overline{R} \cong k[[t]]$, hence it is again local.
Denoting $R^{\mathfrak{m}} = R_1$ we can blow up its maximal ideal and so on, getting, since $\overline{R}$ is a finite R-module:

$$R = R_0 \subset R_1 \subset \cdots \subset R_l = \overline{R} = \overline{R} = \cdots$$

The sequence of multiplicities $e_i = e(R_i)$ is the multiplicity sequence of R.

More generally, if R is a curve and I an ideal of R, the blowing up R^I of I is $\cup_{n>0}(I^n : I^n) = I^{n_0} : I^{n_0}$ for some n_0.

Again we can associate to R a sequence (Lipman sequence) of semilocal rings

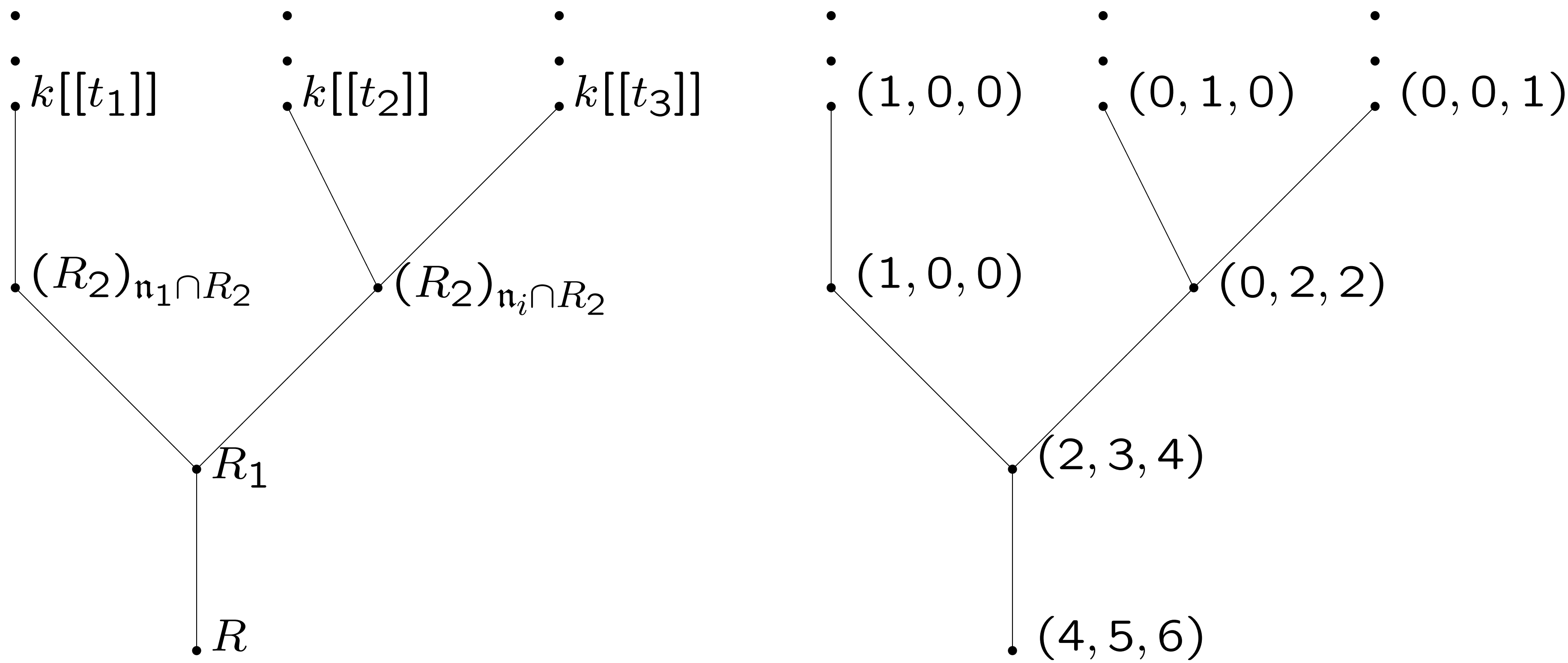
$$R = R_0 \subset R_1 \subset \cdots \subset R_l = \overline{R} = \overline{R} = \cdots$$

where R_{i+1} is obtained from R_i by blowing up the Jacobson radical of R_i, $\mathrm{J}(R_i)$.

Given a maximal ideal n_j of $\overline{R}$ the branch sequence of R along n_j is the sequence of rings $(R_i)_{n_j \cap R_i}$ and the multiplicity sequence of R along n_j is given by the multiplicities of these rings.

Proposition. If $(R, m_1, \ldots, m_r)$ is a Noetherian semilocal ring with $\overline{R} = V_1 \times \cdots \times V_d$, where V_i is a DVR, then $R \simeq R_{m_1} \times \cdots \times R_{m_r}$.

Hence to an algebroid curve R with $\overline{R} = V_1 \times \cdots \times V_d$ we can associate the blowing up tree of R and its multiplicity tree ($\mathfrak{n}_i$ are the maximal ideals of $\overline{R} = k[[t_1]] \times k[[t_2]] \times k[[t_3]]$):



Picture 3

It is possible to chacterize all the trees that can be realized as multiplicity trees of an algebroid curve [Barucci,_,Fröberg].

But in general (for non-plane singularities) it is NOT possible to reconstruct the multplicity tree only by the value semigroup.

7. Apéry set and value semigroups of plane branches: Apéry algortihm

Let $s \in S \subseteq \mathbb{N}$. The Apéry set of S (with respect to s) is:

$$Ap(S,s) = \{x \in S : x - s \notin S\} = \{a_0 = 0 < a_1 < \cdots < a_{s-1} = f(S) + s\}$$

Apery set can be used to charachterize symmetric semigroups, the type of a semigroup, but also to describe the properties of the tangent cone.

Theorem. [Apéry] [Angermüller]
Let R be a plane algebroid branch, $e = e(R)$ and $v(R) = S$.
Set $Ap(S,e) = \{a_0 = 0 < a_1 < a_2 < \ldots < a_{e-1}\}$; then

$$Ap(v(R_1), e) = \{a_0 < a_1 - e < a_2 - 2e < \ldots < a_{e-1} - (e-1)e\}.$$

Using this result it is easy to compute the multiplicity sequence of a plane branch, by its value semigroup and conversely, reconstruct the value semigroup by its multiplicity sequence.

Example. $R = k[[t^4, t^6 + t^7]]$ (char$(k) \neq 2$). Set $S_i = v(R_i)$.

- $v(R) = S = \langle 4, 6, 13 \rangle$, $e(R) = 4$ and $Ap(S, 4) = \{0, 6, 13, 19\}$.

- Apéry's result implies

$$Ap(S_1, 4) = \{0, 2 = 6 - 4, 5 = 13 - 8, 7 = 19 - 12\},$$

which gives $S_1 = \langle 2, 5 \rangle$ and $e_1 = 2$.

- $Ap(S_1, 2) = \{0, 5\}$, so $Ap(S_2, 2) = \{0, 3 = 5 - 2\}$,
which gives $S_2 = \langle 2, 3 \rangle$ and $e_2 = 2$.

- $Ap(S_2, 2) = \{0, 3\}$, so $Ap(S_3, 2) = \{0, 1\}$ and $S_3 = \mathbb{N}$.

Hence the multiplicity sequence of R is $4, 2, 2, 1, \ldots$.

If we start with the multiplicity sequence
$e_0 = 4, e_1 = 2, e_2 = 2, e_3 = 1, \ldots$
we can go backwards in the sequence of blowups:

• we have $e_3 = 1$, so $S_3 = v(R_3) = \mathbb{N}$.

• $e_2 = 2$: determine the Apery set of $\mathbb{N}$ w.r.t. 2: $\{0, 1\}$,
so $Ap(S_2) = \{0, 3 = 1 + 2\}$ and $S_2 = \langle 2, 3\rangle$.

• $e_1 = 2$: determine the Apery set of S_2 w.r.t. 2: $\{0, 3\}$,
so $Ap(S_1, 2) = \{0, 5 = 3 + 2\}$ and $S_1 = \langle 2, 5\rangle$.

• $e_0 = 4$: determine the Apery set of S_1 w.r.t. 4: $\{0, 2, 5, 7\}$,
so $Ap(S, 4) = \{0, 6 = 2 + 4, 13 = 5 + 8, 19 = 7 + 12\}$ and we get
$S = \langle 4, 6, 13, 19\rangle = \langle 4, 6, 13\rangle$.

The reason is that $R = k[[X,Y]]/(F)$; by Weierstrass preparation theorem, can assume F to be of the form $Y^e + \sum_{i=0}^{e-1} c_i(X)Y^i$, where $e = e(R)$.

Setting $x = X + (F)$ and $y = Y + (F)$, we have $R = k[[x,y]] = k[[x]] + k[[x]]y + \cdots + k[[x]]y^{e-1}$, where $v(y) > v(x) = e$.

Blowing up the maximal ideal we obtain
$R_1 = R[y/x] = k[[x, y/x]] = k[[x]]+k[[x]](y/x)+\cdots+k[[x]](y/x)^{e-1}$.

If $Ap(S,e) = \{a_0 = 0 < a_1 < a_2 < \ldots < a_{e-1}\}$, then

$$a_i = v(y^i + \phi_i(x,y))$$

where $\deg_y(\phi) < i$. Set $f_i = y^i + \phi_i$ and call $\{f_0, \ldots, f_{e-1}\}$ an Apéry basis of R.

In the above example: $R = k[[t^4, t^6 + t^7]]$, $x = t^4$, $y = t^6 + t^7$
$Ap(S, 4) = \{0, 6, 13, 19\}$;
$a_1 = 6 = v(y)$, $a_2 = 13 = v(y^2 - x^3)$, $a_3 = 19 = v(y^3 - x^3 y)$.

$R_1 = k[[t^4, t^2 + t^3]]$, $Ap(v(R_1), 4) = \{0, 2, 5, 7\}$, and e.g.
$5 = v((y^2 - x^3)/x^2))$.

Why can we go backwards?

Proposition. [Barucci, _, Fröberg] Let R be a branch. Set $R_1 = R[y/x]$, $e = v(x)$ and $Ap(S_1, e) = \{a'_0, \dots, a'_{e-1}\}$.
Then we can find a minimal set of generators $\{g_0, \dots, g_{e-1}\}$ of R_1 as $k[[x]]$-module, s.t. $v(g_i) = a'_i$, $g_i = (y/x)^i + \psi_i$ (with $\deg(\psi_i) < i$).
Moreover for any such set $\{g_i x^i \mid i = 0, \dots e-1\}$ is an Apéry basis of R.

Algorithm. Given $S \subset \mathbb{N}$ we can apply the Apéry process. If
• at each step we get and ordered Apery set,
• at the end we get $\mathbb{N}$
• the sequence of multiplicities is the multiplicity sequence of a plane branch,
then the semigroup is the value semigroup of a plane branch ($\rightsquigarrow$ explicit conditions for the semigroup).

Example. $S = \{0, 4, 8, 9, 10, 12, 13, 14, 16, \to \dots\}$;
$Ap(S, 4) = \{0, 9, 10, 19\}$.
Hence we get $\{0, 9-4 = 5, 10-8 = 2, 19-12 = 7\}$ which is not ordered.
We get $S_1 = \{0, 2, 4, \to \dots\}$ and in two more steps we get $\mathbb{N} \rightsquigarrow 4, 2, 2, 1, \dots$ that is admissble.
Applying the process backwards we get the semigroup with ordered Apéry set $\{0, 6, 13, 19\}$ of the previous example.

8. Apéry set and value semigroups of plane curves

Let $S \subset \mathbb{N}^h$ and set $\boldsymbol{\delta} = (d_1, \ldots, d_h) \in S$.
The Apéry set of S (with respect to $\boldsymbol{\delta}$) is:

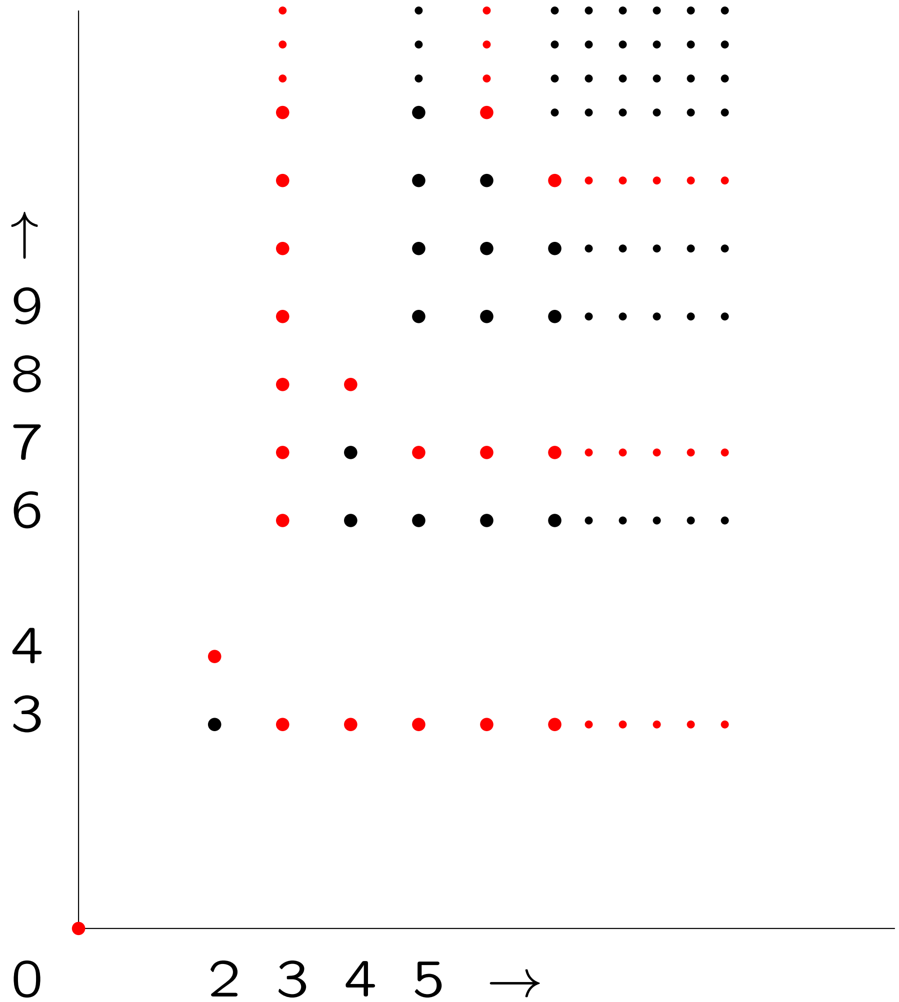
$$Ap(S, \boldsymbol{\delta}) = \{\boldsymbol{\alpha} \in S : \ \boldsymbol{\alpha} - \boldsymbol{\delta} \notin S\}$$

The problem, now, is that $Ap(S, \boldsymbol{\delta})$ is infinite and not linearly ordered.

We would like to have a partition of $Ap(S, \boldsymbol{\delta})$ in $D = d_1 + \cdots + d_h$ subsets:

$$Ap(S, \boldsymbol{\delta}) = \bigcup_{i=0}^{D-1} A_i$$

in such a way that the A_i play the role of the a_i.



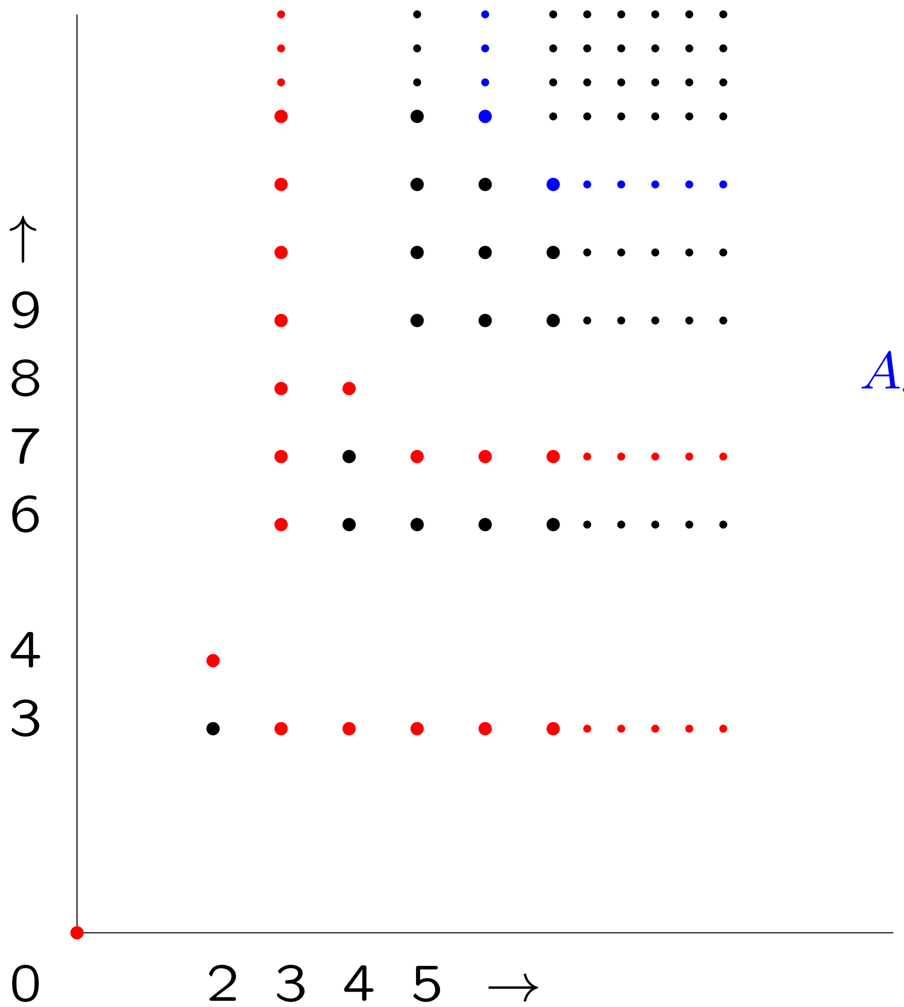
$\boldsymbol{\delta} = (2, 3) = \boldsymbol{e}$

now $Ap(S, \boldsymbol{\delta})$ is infinite

Picture 4. $Ap(S, \boldsymbol{\delta})$

How do we define the A_i?

Define $\boldsymbol{\alpha} \leq\leq \boldsymbol{\beta}$ iff either $\boldsymbol{\alpha} = \boldsymbol{\beta}$ or $\alpha_i < \beta_i$ for both $i = 1, 2$.

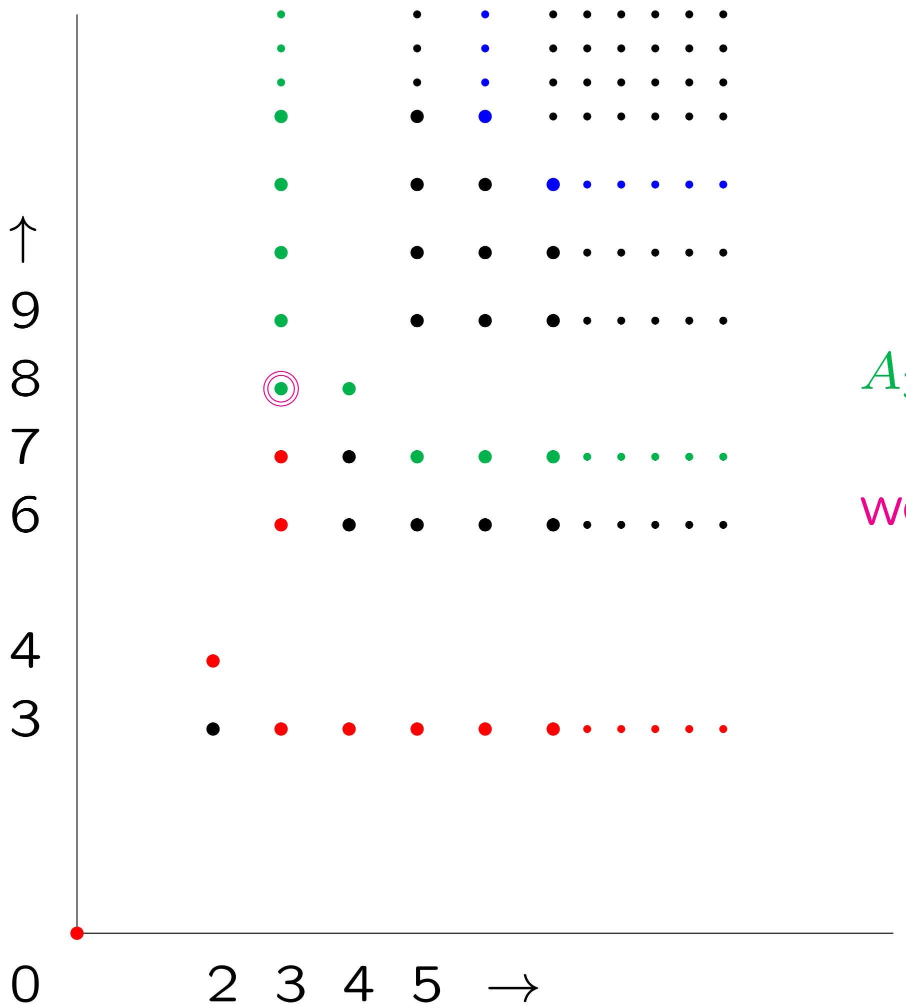


$$\boldsymbol{e} = (2,3), \quad E = 2+3 = 5$$

$$A_4 = \{\boldsymbol{\alpha} \in Ap(S, \boldsymbol{e}) \mid \boldsymbol{\alpha} \text{ max. w.r.t. } \leq\leq\}$$

$$= \Delta^S(\gamma + \boldsymbol{e})$$

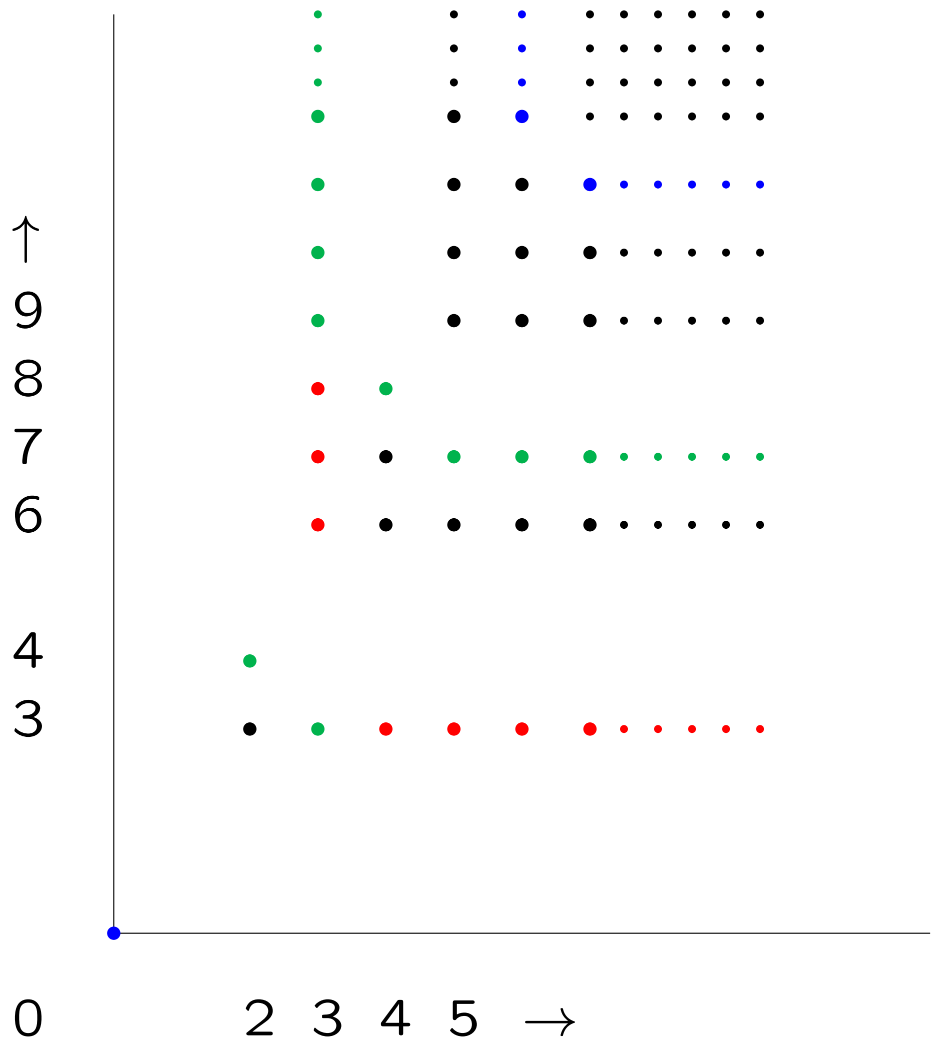
Picture 5. A_4



$A_3 \subseteq \{\boldsymbol{\alpha} \in Ap(S,\boldsymbol{e}) \setminus A_5$ max. w.r.t. $\leq\leq\}$

we exclude the $\boldsymbol{\beta}$ obtained as infimums

Picture 6. A_3



Picture 7. $Ap(S, \boldsymbol{e}) = A_0 \cup A_1 \cup A_2 \cup A_3 \cup A_4$

Theorem [_, Guerrieri, Micale] [Guerrieri, Maugeri, Micale] Let $S \subseteq \mathbb{N}^h$ be a good semigroup, $\boldsymbol{\delta} = (d_1, \ldots, d_h)$ and $D = d_1 + \cdots + d_h$. Then $Ap(S, \boldsymbol{\delta}) = \cup_{i=0}^{D-1} A_i$.

Remark. We can construct the partition for any complement of a good ideal I and the number of levels measures the distance between S and I:

Theorem. [_, Guerrieri, Micale] [Guerrieri, Micale, Maugeri] Let $S \subseteq \mathbb{N}^h$ be a good semigroup. Let $A \subseteq S$ such that $I := S \setminus A$ is a proper good ideal of S and let $A = \bigcup_{i=0}^{N-1} A_i$ be the partition of A. Then

$$N = d(S \setminus I).$$

9. Apéry algorithm for plane curves

Let $R = k[[X,Y]]/(F)$; with $F = G_1 G_2 \cdots G_h$, (G_i irreducible, pairwise distinct).
By Weierstrass preparation thm. and up to a change of variables, we can assume $F = Y^E + \sum_{i=0}^{E-1} c_i(X)Y^i$, where $E = e(R)$.
Setting $x = X + (F)$ and $y = Y + (F)$, we have:

$R = k[[x,y]] = k[[x]] + k[[x]]y + \cdots + k[[x]]y^{E-1}$,
where $v(y) > v(x) = \boldsymbol{e} = (e_1, \ldots, e_h)$ and $E = e_1 + \cdots + e_h$.

Set: $U_i = k[[x]] + k[[x]]y + \cdots + k[[x]]y^i \quad \forall\ 0 \leq i \leq E-1$,
$T_0 = \{1\}$ and

$$T_i = \{y^i + \phi_i(x,y) \mid \phi_i(x,y) \in U_{i-1}, v(y^i + \phi_i(x,y)) \notin v(U_{i-1})\}.$$

Theorem. [Barucci, _ , Fröberg] Setting $Ap(v(R), \boldsymbol{e}) = \cup_{i=0}^{E-1} A_i$, then $A_i = v(T_i)$.

Blowing up the maximal ideal we obtain
$R_1 = R[y/x] = k[[x]] + k[[x]](y/x) + \cdots + k[[x]](y/x)^{E-1}$.

When R_1 is still local we can go backwards.

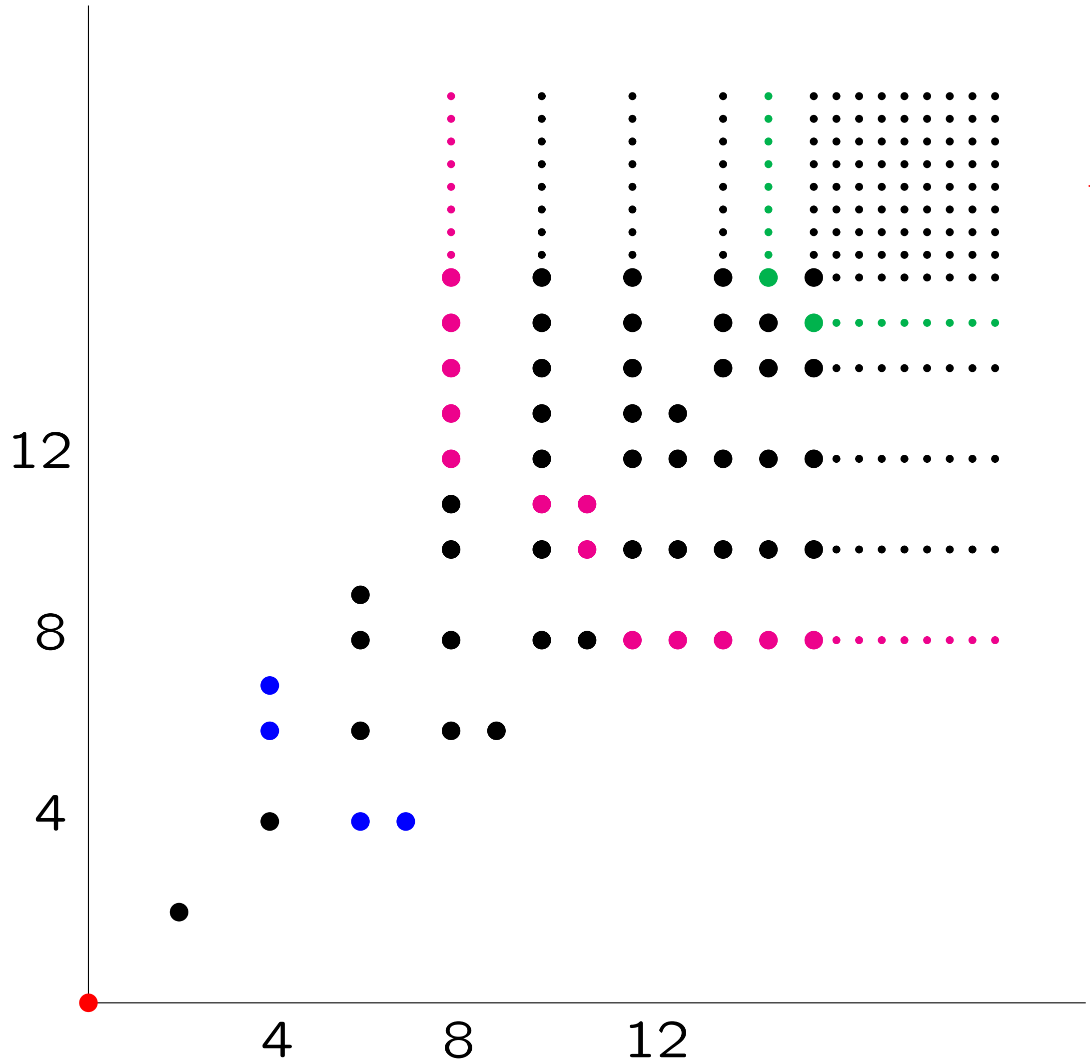
Theorem. [Barucci, _, Fröberg] Let R be a plane algebroid curve and assume $R_1 = R[y/x]$ local. Let $\boldsymbol{e} = v(x) = (e_1, \ldots, e_h)$ and $E = e_1 + \cdots + e_h$. Set $Ap(S, \boldsymbol{e}) = \cup_{i=0}^{E-1} A_i$ and $Ap(S_1, \boldsymbol{e}) = \cup_{i=0}^{E-1} A'_i$. Then , $\forall i$, $A_i = A'_i + i\boldsymbol{e}$.

The reason is that $g = y/x \in R_1$ is such that

$$R_1 = k[[x]] + k[[x]]g + \cdots + k[[x]]g^{e-1}$$

and $\forall i$, $A'_i = \{v(g^i + \psi_i) \mid \ldots\}$.

Remark. We are using the presentation of R_1 as quotient of $k[[X, Y]]$.



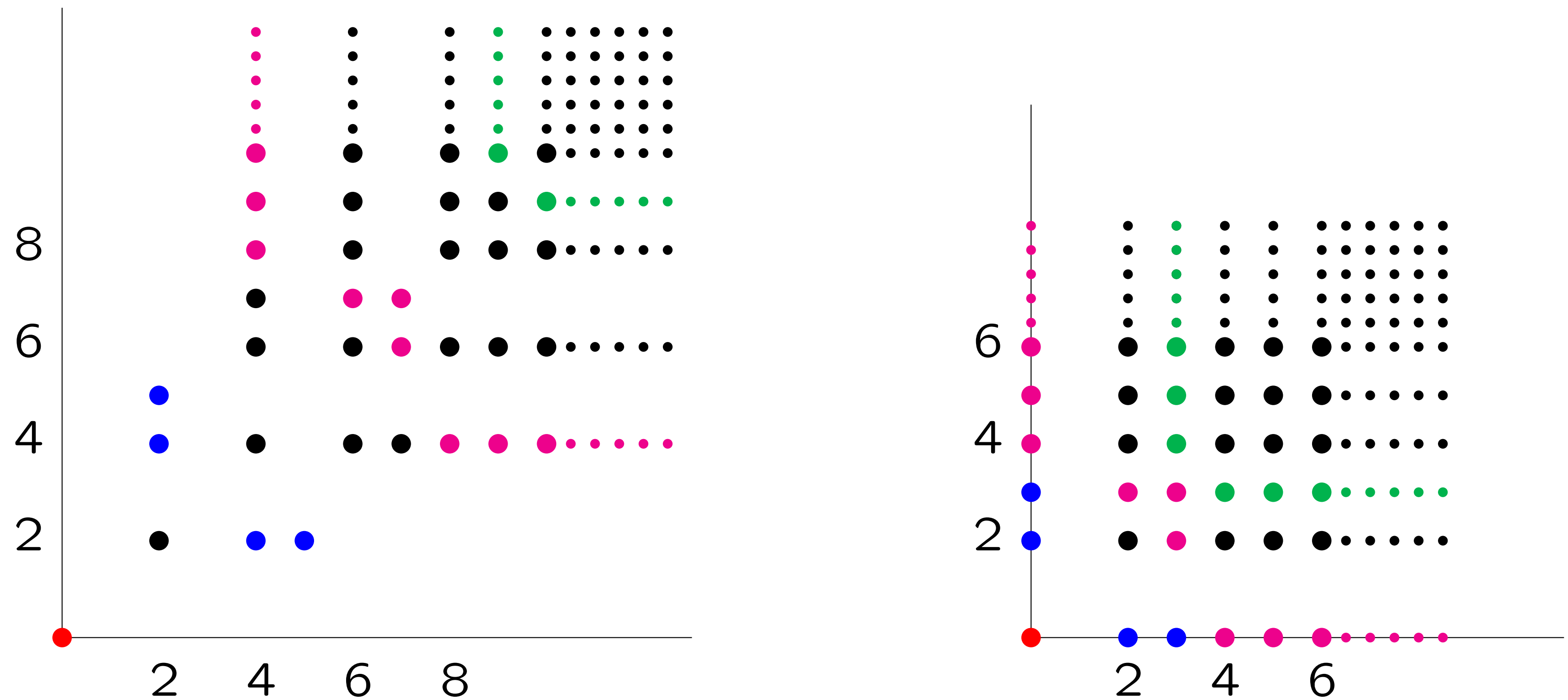
$$R = \frac{k[[x,y]]}{(x^7-y^2)\cap(x^7-x^4+2x^2y-y^2)}$$

$$x \mapsto (t^2, u^2)$$

$$y \mapsto (u^7, u^4+u^7)$$

$$\boldsymbol{e} = (2,2)$$

Picture 8. $S = v(R)$ $Ap(S,\boldsymbol{e}) = A_0 \cup A_1 \cup A_3 \cup A_4$



Picture 9. $Ap(v(R_1, \boldsymbol{e})) = A_0 \cup A_1 \cup A_3 \cup A_4$ and $Ap(v(R_2), \boldsymbol{e}_1))$

Now $R_2 \cong R_{2,1} \times R_{2,2}$ is semilocal and $S_2 := v(R_2) = \pi_1(S_2) \times \pi_2(S_2)$.

Once we are in the semilocal case we can proceed the blowing up process, working on the localizations. So we can go on from R to $\bar{R}$ and compute the multiplicity tree by the semigroup.
If, conversely, we want to obtain the semigroup from the multiplicity tree, the problem arise passing backwards from the non-local to the local case. More precisely we need:

$\mathbf{1}$. describe the levels of the Apéry set in function of the levels of the projections;

$\mathbf{2}$. find a description of $R = R_1 \times R_2$ as a $k[[f]]$-module, $f = (f_1, f_2) \in R$, genererated by the powers of another element $g = (g_1, g_2)$;

$\mathbf{3}$. Find an analogue of the results that charachterize the levels of the Apéry set w.r.t. $v(f)$ as value sets, depending on the power of g.

Problem 1. was solved completely (i.e. for any $h \geq 2$) [Guerrieri, Maugeri, Micale].

As for problems 2. and 3. we have the solution for $h = 2$:

Theorem. [_, Delgado, Guerrieri, Maugeri, Micale] Let $W = W_1 \times W_2$ be a non local ring, $\overline{W} = k[[t]] \times k[[u]]$. Let $S = v(W)$, fix $\boldsymbol{\epsilon} = (\epsilon_1, \epsilon_2) \in S$, with $\epsilon_1, \epsilon_2 > 0$ and set $E = \epsilon_1 + \epsilon_2$. Choose any $f = (f_1, f_2) \in W$ of value $v(f) = \boldsymbol{\epsilon}$. Then there exists $g = (g_1, g_2) \in W$, s.t. $W = k[[f]] + k[[f]]g + \cdots + k[[f]]g^{E-1}$.

Theorem. [_, Delgado, Guerrieri, Maugeri, Micale] Set $U_i = k[[f]] + k[[f]]g + \cdots + k[[f]]g^i$ for any $i = 0, \ldots, E-1$, $T_0 = \{1\}$ and

$$T_i = \{y^i + \phi_i(x, y) \mid \phi_i(x, y) \in U_{i-1}, v(y^i + \phi_i(x, y)) \notin v(U_{i-1})\}.$$

Then, setting $Ap(v(W), \boldsymbol{\epsilon}) = \cup A'_i$, $A'_i = v(T_i)$.

Corollary. If $W = R_1$,
set $Ap(S, \boldsymbol{e}) = \cup_{i=0}^{E-1} A_i$ and $Ap(S_1, \boldsymbol{e}) = \cup_{i=0}^{E-1} A'_i$.
Then , $\forall i$,

$$A_i = A'_i + i\boldsymbol{e}.$$

Hence to give a semigroup of a plane curve with two branches is equivalent to give its multiplicity tree.

In [Barucci, _, Fröberg] we characterized the multiplicity trees of plane curves with two branches.

Thus we can give an algorithm to check if a good semigroup is the value semigroup of a plane singularity with two branches.

What does remain to do? Since the general (non local case) can be studied looking at R as $R_1 \times R_2$ (with R_i either local or not), in order to get the complete solution ($h \geq 3$), we can proceed by induction on the number of branches, but we still have to solve some technical problems.
Moreover we have to give and explicit description of the admissible multiplicity trees for a plane singularity with h branches.

THANKS FOR YOUR ATTENTION!