Numerical Semigroups and Music

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Conference on Rings and Factorizations 2023,
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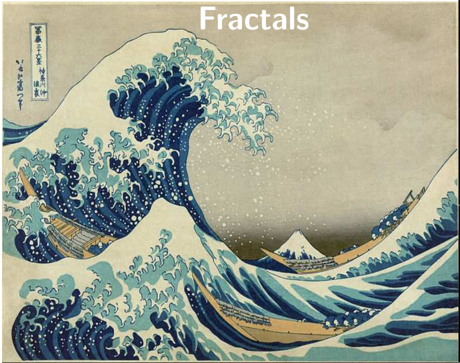
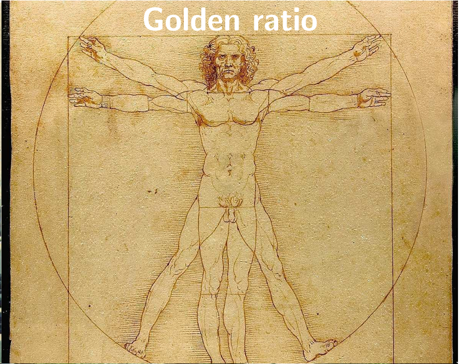
Fractals



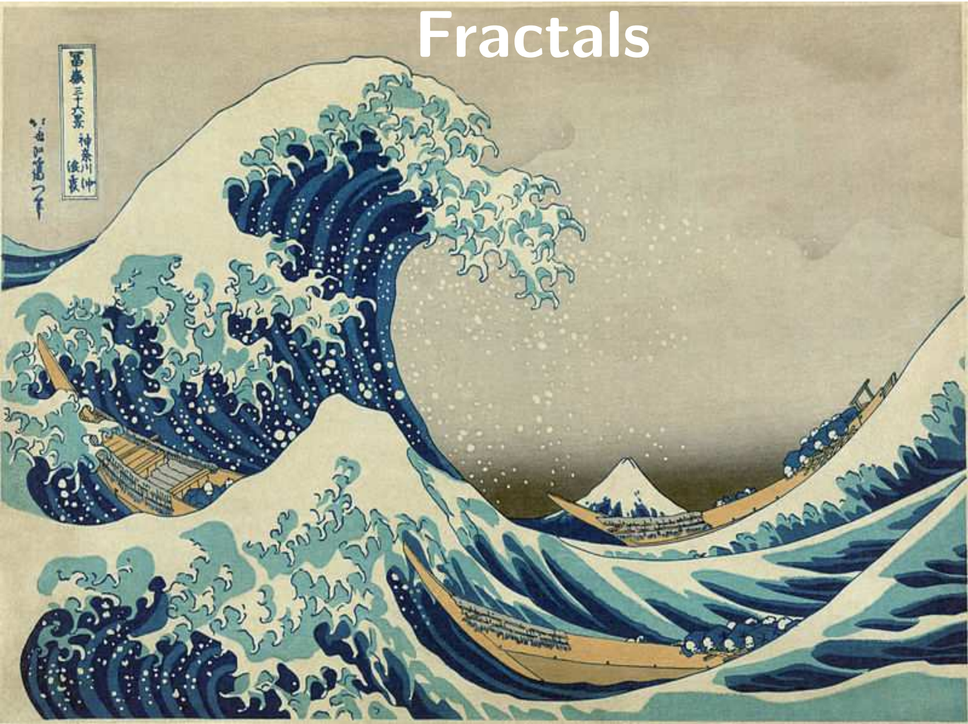
Golden ratio



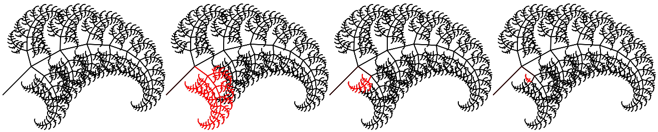
Harmonics and semigroups

Tempered monoids

Fractals



Fractals

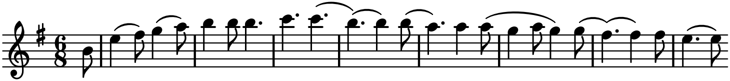


Fractals in music

Bedřich Smetana

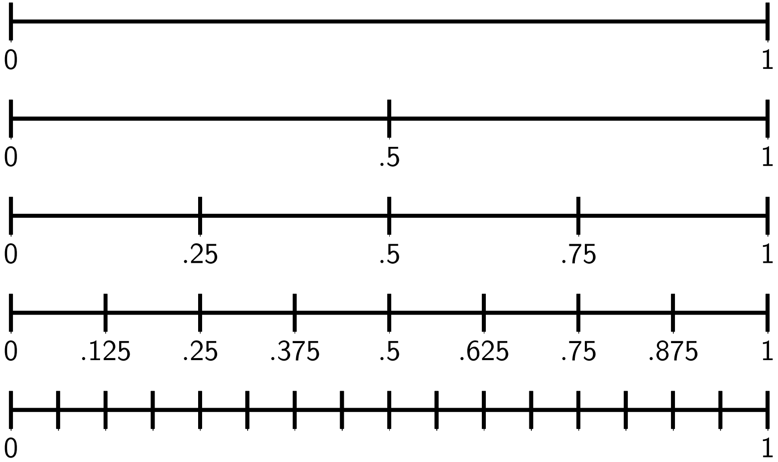
Moldau (Mein Vaterland)

Big wave

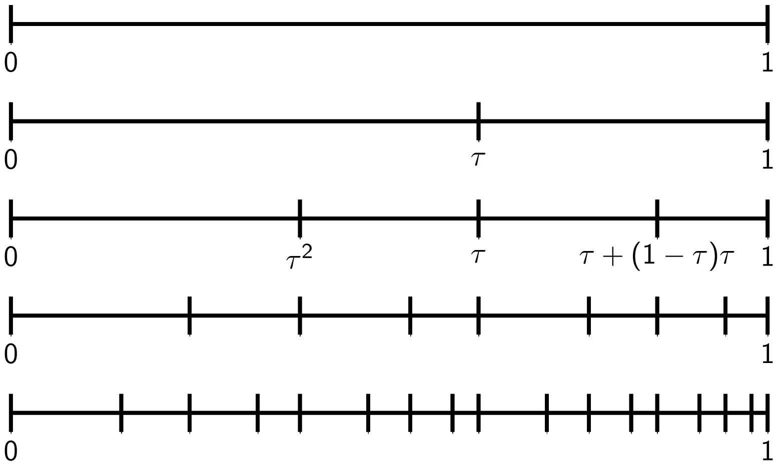


Smaller and smaller waves

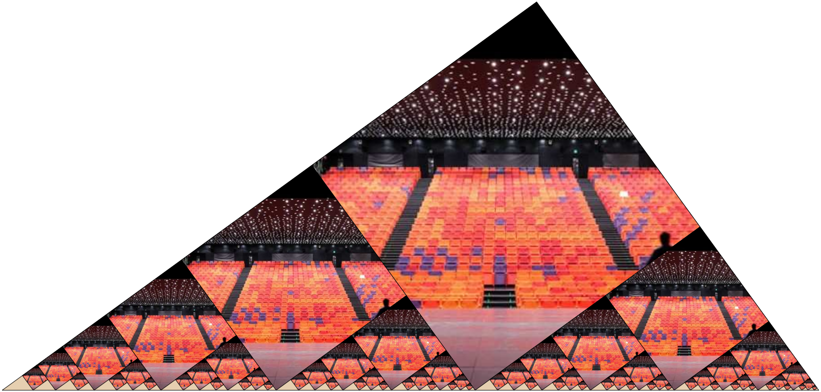
Fractal division of an interval



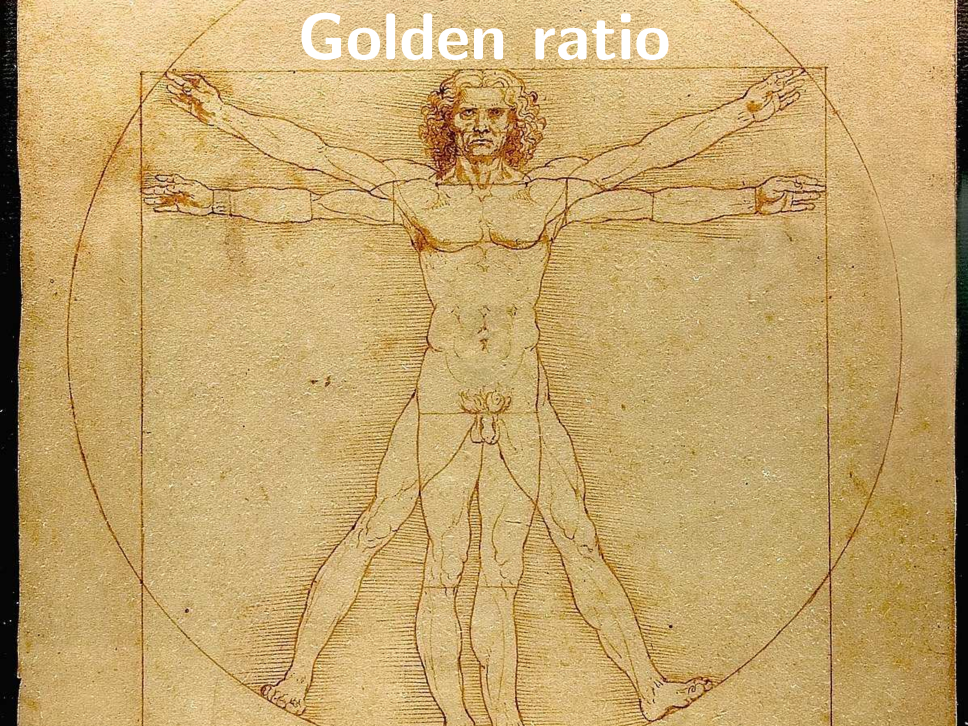
Fractal division of an interval: nonbisectional



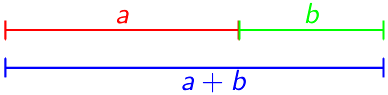
Fractal division of an interval seen in a fractal



Golden ratio

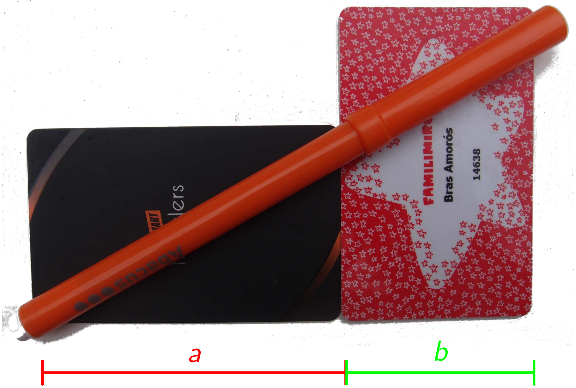


The golden ratio

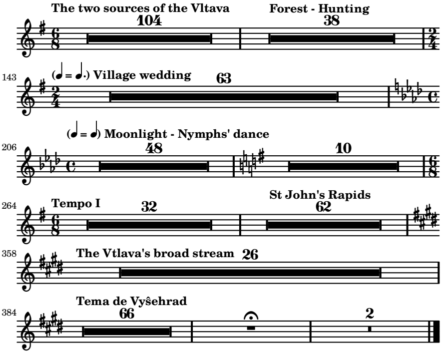


$$\frac{b}{a} = \frac{a}{a+b} = \varphi$$

The golden ratio in your wallet



The golden ratio in music



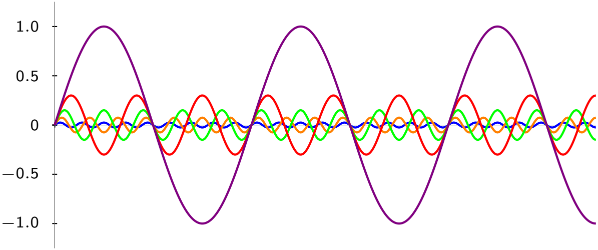
Let's count beats:

				cumulative count	portion
	104	×	2	208	
+	38	×	2	284	
+	63	×	2	410	φ
+	48	×	4	602	
+	10	×	4	642	
Tempo I					
+	32	×	2	706	
+	62	×	2	830	
+	26	×	2	882	$1-\varphi$
+	66	×	2	1014	
+	1	×	2	1016	
+	2	×	2	1020	
	1020				

Harmonics and semigroups

Harmonics

Harmonics



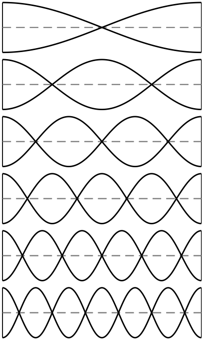
Frquency indicates the pitch: $la = 440Hz$.
Frequency of harmonics: multiples of $440Hz$ ($880Hz$, $1320Hz$, $1760Hz$, etc.)

Harmonics: different models

cylindrical open pipe	cylindrical closed pipe	string
		

cylindrical open pipe

motion of the air



$\nu = \nu_0$, $\lambda_0 = 2L$

$\nu_1 = 2\nu_0$, $\lambda_1 = L$

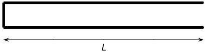
$\nu_2 = 3\nu_0$, $\lambda_2 = 2L/3$

$\nu_3 = 4\nu_0$, $\lambda_3 = L/2$

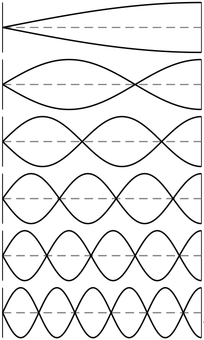
$\nu_4 = 5\nu_0$, $\lambda_4 = 2L/5$

$\nu_5 = 6\nu_0$, $\lambda_5 = L/3$

cylindrical closed pipe



motion of the air



$\nu = \nu_0$, $\lambda_0 = 4L$

$\nu_1 = 3\nu_0$, $\lambda_1 = 4L/3$

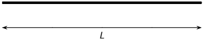
$\nu_2 = 5\nu_0$, $\lambda_2 = 4L/5$

$\nu_3 = 7\nu_0$, $\lambda_3 = 4L/7$

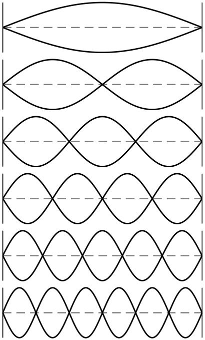
$\nu_4 = 9\nu_0$, $\lambda_4 = 4L/9$

$\nu_5 = 11\nu_0$, $\lambda_5 = 4L/11$

string



string vibration



$\nu = \nu_0$, $\lambda_0 = 2L$

$\nu_1 = 2\nu_0$, $\lambda_1 = L$

$\nu_2 = 3\nu_0$, $\lambda_2 = 2L/3$

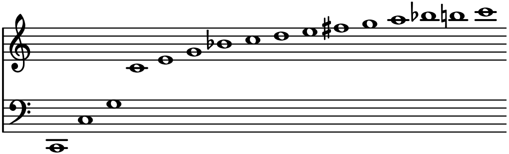
$\nu_3 = 4\nu_0$, $\lambda_3 = L/2$

$\nu_4 = 5\nu_0$, $\lambda_4 = 2L/5$

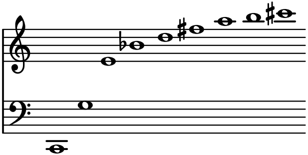
$\nu_5 = 6\nu_0$, $\lambda_5 = L/3$

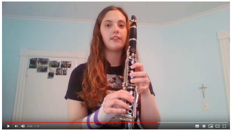
Harmonics: different models

Strings and cylindrical open pipes:



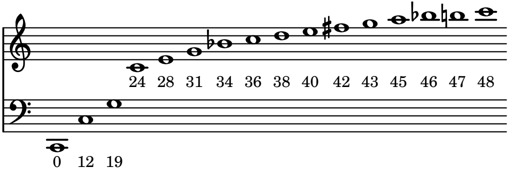
Cylindrical pipes with one open end and one closed end:





Harmonics: 12-semitone count

Divide the octave into 12 equal semitones.
What semitone interval corresponds to each harmonic?



$$
\begin{array}{rl}
H_{open} = & \{0,12,19,24,28,31,34,36,38,40,42,43,45,46,47,48,49,50,51,52,53,54,55,56,\ldots\} \\
& \;\updownarrow \quad\;\; \updownarrow \quad\;\; \updownarrow \quad\;\; \updownarrow \quad\;\; \updownarrow \quad\;\; \updownarrow \quad\;\; \updownarrow \quad\;\; \updownarrow \quad\;\; \updownarrow \quad\;\; \updownarrow \quad\;\; \updownarrow \quad\;\; \updownarrow \; \updownarrow \\
H_{closed} = & \{0, \quad 19, \quad 28, \quad 34, \quad 38, \quad 42, \quad 45, \quad 47, \quad 49, \quad 51, \quad 53, \quad 55,56,\ldots\}
\end{array}
$$

Other equal divisions of the octave

Instruments such as the gamelan divide the octave into more than 12 parts.

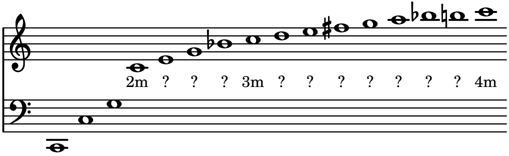
Compositions exist in $19, 24, 31$ and other equal temperaments.

Jeffrey Harrington
Prelude 3
19ET

Fabio Costa
Aphoristic Madrigal
31ET

Harmonics: m-semitone count

Divide the octave into m equal semitones.
What semitone interval corresponds to each harmonic?



$$H_{open} = \{0, m, ?, 2m, ?, ?, ?, 3m, ?, ?, ?, ?, ?, ?, ?, 4m, ?, ?, ?, ?, ?, ?, ?, ?, \ldots\}$$

$$H_{closed} = \{0, \quad ?, \quad ?, \quad ?, \quad ?, \quad ?, \quad ?, \quad ?, \quad ?, \quad ?, \quad ?, \quad ?, \quad \ldots\}$$

Numerical semigroups

Natural properties of H_{open} and H_{closed}:

- H contains 0
- H has only a finite number of gaps
- H is closed under addition
  - overtones of an overtone of a fundamental tone should be overtones of that fundamental tone
  - if $a \in H$ and $b \in H$, then $a + b \in H$.

} numerical semigroup

Logarithms, amplifying and discretizing

i	$\log_2(i)$	$12\log_2(i)$	$[12\log_2(i)]$	$[12\log_2(i)]_{.4}$
1	0	0	0	0
2	1	12	12	12
3	1.58	19.02	19	19
4	2	24	24	24
5	2.32	27.86	28	28
6	2.58	31.02	31	31
7	2.81	33.69	34	34
8	3	36	36	36
9	3.17	38.04	38	38
10	3.32	39.86	40	40
11	3.46	41.51	42	42
12	3.58	43.02	43	43
13	3.70	44.41	44	45
14	3.81	45.69	46	46
15	3.91	46.88	47	47
16	4	48	48	48

Tempered monoids

Tempered monoids

The logarithm sequence L satisfies

- $0 \in L$
- L is closed under addition
  - If $a = \log_2(i)$ and $b = \log_2(j)$, then $a + b = \log_2(i \times j)$
- Its terms get indefinitely close

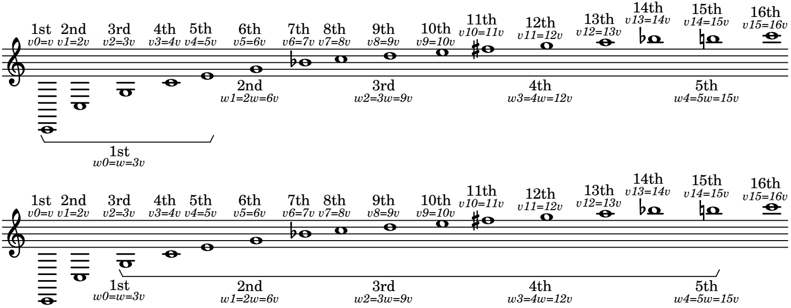
Tempered monoid

Tempered monoids

Tempered monoids in $\mathbb{R}$	Numerical semigroups in $\mathbb{N}$
$0 \in M$	$0 \in H$
M closed under addition	H closed under addition
M terms get indefinitely close	H has finite number of gaps

Recall $H = [12L]_{0.4}$.
If the first non-zero term of M is 1,
M tempered monoid $\Longrightarrow [mM]_\alpha$ likely to be a numerical semigroup
with first non-zero term m.

Product-compatible monoid



pitch (5th harm.) - pitch (fundamental) = pitch (5th harm. of 3rd harm) - pitch (3rd harm.)

$$\text{pitch}(a) - 0 = \text{pitch}(a \cdot b) - \text{pitch}(b)$$

$$\text{pitch}(a \cdot b) = \text{pitch}(a) + \text{pitch}(b)$$

Product-compatible monoids

Theorem: The unique product-compatible monoid is the logarithm sequence.

Proof

Suppose $M = \mu_1, \mu_2, \ldots$ with $\mu_i < \mu_{i+1}$ and $\mu_2 = 1$.

$$\mu_{ab} = \mu_a + \mu_b \Longrightarrow \mu_{a^j} = j\mu_a.$$

Claim: $\mu_i = \log_2(i)$ for any $i \in \mathbb{N}_0$.
Otherwise, $\mu_i < \log_2(i)$ (similarly if $\mu_i > \log_2(i)$) and

$$\mu_i < p/q < \log_2(i).$$

Then

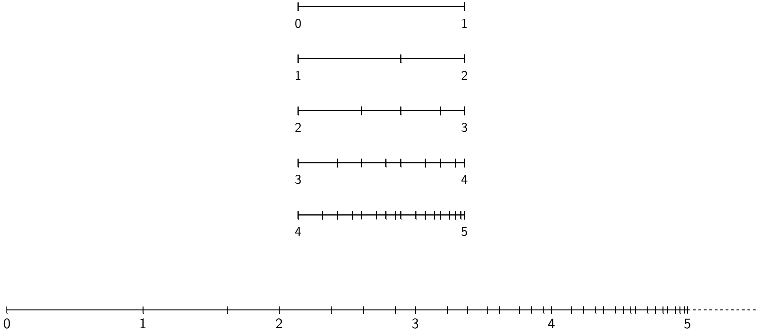
$$\mu_{i^q} = q\mu_i < p = p\mu_2 = \mu_{2^p},$$

but

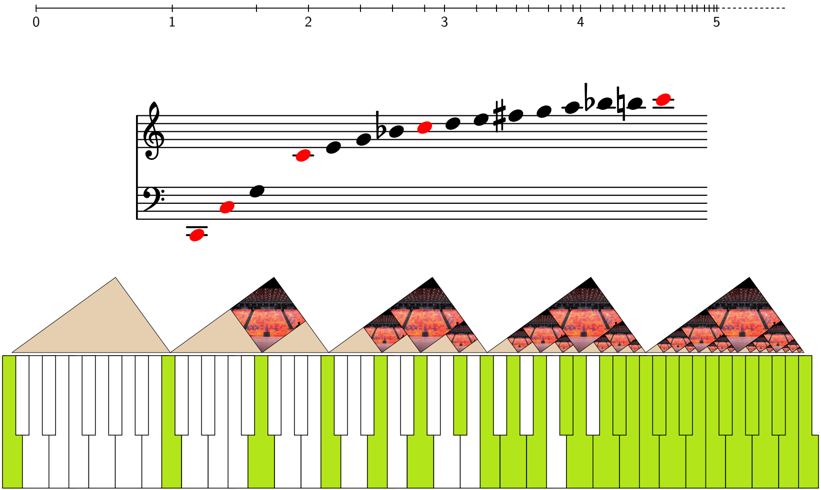
$$i^q = 2^{q \log_2(i)} > 2^p,$$

a contradiction.

Fractal monoids

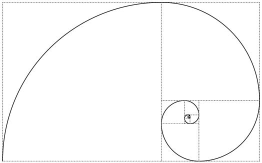


Fractal monoids



Fractal monoids

Theorem: The unique non-bisectional fractal monoid is the one given by dividing intervals by the golden ratio.



Proof

1. The period $\{1, 1+\varphi\}$ generates a fractal monoid (DIFFICULT)
2. The unique non-bisectional normalized monoid is exactly the fractal monoid generated by $\{1, 1+\varphi\}$. (EASY)

The first periods of the monoid must be (for some $p \neq 0.5$)
$\{0\}$,
$\{1, 1+p\}$,
$\{2, 2+p^2, 2+p, 2+2p-p^2\}$,
$\{3, 3+p^3, 3+p^2, 3+2p^2-p^3, 3+p, 3+p+p^2-p^3, 3+2p-p^2, 3+3p-3p^2+p^3\}$,

$$(1+p)+(1+p) = 2+2p \in M.$$

If $p < 0.5 \Longrightarrow 2+p < 2+2p < 3 \Longrightarrow 2+2p = 2+2p-p^2 \Longrightarrow p = 0.$
If $p > 0.5$, then $3 < 2+2p < 3+p$. Then either

- $2+2p = 3+p^3 \Longrightarrow p^3-2p+1 = (p^2+p-1)(p-1) = 0.$
  Positive solutions: $p = 1$ and $p = \frac{-1+\sqrt{5}}{2} = \varphi.$
- $2+2p = 3+p^2 \Longrightarrow p^2-2p+1 = (p-1)^2 = 0 \Longrightarrow p = 1.$
- $2+2p = 3+2p^2-p^3 \Longrightarrow p^3-2p^2+2p-1 =$
  $(p^2-p+1)(p-1) = 0 \Longrightarrow p = 1$ or $p \notin \mathbb{R}$

Simultaneous discretization

$\log_2(i)$	$12\log_2(i)$	$[12\log_2(i)]_{0.40}$	$[12F_i]_{1.00}$	$12F_i$	F_i
0	0	0	0	0	0
1	12	12	12	12	1
1.58	19.02	19	19	19.42	1.62
2	24	24	24	24	2
2.32	27.86	28	28	28.58	2.38
2.58	31.02	31	31	31.42	2.62
2.81	33.69	34	34	34.25	2.85
3	36	36	36	36	3
3.17	38.04	38	38	38.83	3.24
3.32	39.86	40	40	40.58	3.38
3.46	41.51	42	42	42.33	3.53
3.58	43.02	43	43	43.42	3.62
3.70	44.41	45	45	45.17	3.76
3.81	45.69	46	46	46.25	3.85
3.91	46.88	47	47	47.33	3.94
4	48	48	48	48	4

Simultaneous discretization

But $m = 12$ is not the unique option

$\log_2(i)$	$10\log_2(i)$	$[10\log_2(i)]_{0.50}$	$[10F_i]_{1.00}$	$10F_i$	F_i
0	0	0	0	0	0
1	10	10	10	10	1
1.58	15.85	16	16	16.18	1.62
2	20	20	20	20	2
2.32	23.22	23	23	23.82	2.38
2.58	25.85	26	26	26.18	2.62
2.81	28.07	28	28	28.54	2.85
3	30	30	30	30	3
3.17	31.70	32	32	32.36	3.24
3.32	33.22	33	33	33.82	3.38
3.46	34.59	35	35	35.28	3.53
3.58	35.85	36	36	36.18	3.62
3.70	37.00	37	37	37.64	3.76
3.81	38.07	38	38	38.54	3.85
3.91	39.07	39	39	39.44	3.94
4	40	40	40	40	4

Simultaneous discretization

But $m = 12$ is not the unique option

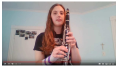
$\log_2(i)$	$13\log_2(i)$	$[13\log_2(i)]_{0.18}$	$[13F_i]_{0.94}$	$13F_i$	F_i
0	0	0	0	0	0
1	13	13	13	13	1
1.58	20.60	21	21	21.03	1.62
2	26	26	26	26	2
2.32	30.19	31	31	30.97	2.38
2.58	33.60	34	34	34.03	2.62
2.81	36.50	37	37	37.10	2.85
3	39	39	39	39	3
3.17	41.21	42	42	42.07	3.24
3.32	43.19	44	44	43.97	3.38
3.46	44.97	45	45	45.86	3.53
3.58	46.60	47	47	47.03	3.62
3.70	48.11	48	48	48.93	3.76
3.81	49.50	50	50	50.10	3.85
3.91	50.79	51	51	51.28	3.94
4	52	52	52	52	4

Simultaneous discretization

But $m = 12$ is not the unique option

$\log_2(i)$	$18\log_2(i)$	$[18\log_2(i)]_{0.05}$	$[18F_i]_{0.88}$	$18F_i$	F_i
0	0	0	0	0	0
1	18	18	18	18	1
1.58	28.53	29	29	29.12	1.62
2	36	36	36	36	2
2.32	41.79	42	42	42.88	2.38
2.58	46.53	47	47	47.12	2.62
2.81	50.53	51	51	51.37	2.85
3	54	54	54	54	3
3.17	57.06	58	58	58.25	3.24
3.32	59.79	60	60	60.88	3.38
3.46	62.27	63	63	63.50	3.53
3.58	64.53	65	65	65.12	3.62
3.70	66.61	67	67	67.75	3.76
3.81	68.53	69	69	69.37	3.85
3.91	70.32	71	71	71.00	3.94
4	72	72	72	72	4

Simultaneous discretization

Why not using, then, $m = 13$ or $m = 18$? Recall: 

$\log_2(i)$	$13\log_2(i)$	$[13\log_2(i)]_{0.18}$	$[13F_i]_{0.94}$	$13F_i$	F_i
0	0	0	0	0	0
1	13	13	13	13	1
1.58	20.60	21	21	21.03	1.62
2	26	26	26	26	2
2.32	30.19	31	31	30.97	2.38
2.58	33.60	34	34	34.03	2.62
2.81	36.50	37	37	37.10	2.85
3	39	39	39	39	3
3.17	41.21	42	42	42.07	3.24
3.32	43.19	44	44	43.97	3.38
3.46	44.97	45	45	45.86	3.53
3.58	46.60	47	47	47.03	3.62
3.70	48.11	48	48	48.93	3.76
3.81	49.50	50	50	50.10	3.85
3.91	50.79	51	51	51.28	3.94
4	52	52	52	52	4

Conclusion

Multiplicity 12 gives the largest value for which the discretization of the unique product-compatible monoid keeps the property that every interval is successively divided using the same ratio (in fact, the golden ratio), and also, it satisfies the property of being half-closed-pipe admissible.

Reference:

M. Bras-Amorós: *Tempered Monoids of Real Numbers, the Golden Fractal Monoid, and the Well-Tempered Harmonic Semigroup*, Semigroup Forum, Springer, vol. 99, n. 2, pp. 496-516, November 2019. ISSN: 0037-1912.

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