Arithmetic of minimal factorizations

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Basic definitions

H = (multiplicative) monoid with identity 1_H.
$H^\times$ = group of units or invertible elements of H.

The monoid H is

- **cancellative** if $ax = bx$ or $xa = xb \Rightarrow a = b$, for every $a, b, x \in H$;
- **unit-cancellative** if $a = ax$ or $a = xa \Rightarrow x \in H^\times$, for every $a, x \in H$.

A **non-unit** element $a \in H$ is

- an **atom** of H if $a = xy \Rightarrow x \in H^\times$ or $y \in H^\times$.
- an **irreducible** of H if $a = xy \Rightarrow x \in HaH$ or $y \in HaH$.

Note: atom $\Rightarrow$ irreducible, but in general irreducible $\not\Rightarrow$ atom.

E.g., if D is a domain, $0 \in D$ is irreducible but not an atom.

The two notions are equivalent if H is commutative and unit-cancellative

The classical theory of factorization

A great variety of problems from all areas of mathematics involve the **decomposition** of *certain* elements of a monoid in terms of *certain* other elements that act as **building blocks** and, in a sense, cannot be broken down into smaller pieces.

In the "classical theory of factorization", the building blocks are *mainly* atoms and two are the basic questions addressed:

1. Check if every non-unit of H factors as a product of atoms (i.e., H is **atomic**); and if not, characterize which elements do.
2. Assuming that H is atomic, **qualify and quantify** by the use of "invariants" the non-uniqueness, however defined, of the factorizations.

Answering these questions when H is **commutative and cancellative** has led to a solid theory, *partly* extended recently to the unit-cancellative case.

A. GEROLDINGER, F. HALTER-KOCH, Non-Unique Factorizations, 2006

Moving away from the classical setting

Moving from a commutative to a non-commutative setting the extensions of the classical theory work nicely if there exists a **“transfer morphism”** from H to a commutative and [unit-]cancellative monoid, otherwise the theory shows **some “gaps”**:

1. some monoids are **not atomic** even if they should morally be (e.g., H is cancellative, reduced, and 2-generated);
2. the classical invariants associated with atomic factorizations (or with decompositions defined in terms of a different type of building blocks) **blow up** in a predictable way and lose most of their significance.

Analogous issues appear in highly non-cancellative (even commutative) monoids, e.g., in the presence of non-trivial idempotents or in rings with non-zero zero divisors.

The above gaps have been recognized for a long time and various solutions have been proposed to overcome their effects. E.g., for commutative rings with zero divisors *irreducibles* (instead of atoms) are the building blocks that extend the classical results.

D. D. ANDERSON, S. VALDES-LEON, Factorization in Commutative Rings with Zero Divisors, *Rocky Mount. J. Math*, 1996.

Our strategy

Fundamental aspects of the classical theory of factorization can be widely generalized by combining the languages of **monoids** and **preorders**.

Endowing H with a suitable preorder permits to define a general notion of *irreducible* suitable for the extension of classical results on atomicity.

In addition, an appropriate preorder on the free monoid $\mathscr{F}(H)$ over H allows the introduction of *minimal factorizations*, a refinement of classical factorizations that counter the blow-up phenomena mentioned before.

A **preorder** on a set X is a *reflexive* and *transitive* (binary) relation $\preceq$ (read *"pē"*) on X. We say that x and y are **$\preceq$-equivalent** if $x \preceq y \preceq x$, and we write $x \prec y$ to mean that $x \preceq y$ and $y \not\preceq x$.

A preorder induced by the multiplication of a monoid H is the divisibility preorder $|_H$: for all $x, y \in H$, $x \mid_H y$ if and only if $y = uxv$ for some $u, v \in H$.

Premonoids

Definition (Premons and $\preceq$-irreds)

Let H be a monoid and $\preceq$ be a preorder on its underlying set. The couple $\mathcal{H} := (H, \preceq)$ is said to be a **premonoid** (or **premon**, for short).

An element $u \in H$ is a **$\preceq$-unit** (of H) if $u \preceq 1_H \preceq u$, and is a **$\preceq$-non-unit** otherwise.

We say that a $\preceq$-non-unit $x \in H$ is a **$\preceq$-irreducible** (or **$\preceq$-irred**) (of H) if $x \neq y_1 y_2$ for all $\preceq$-non-units $y_1 \prec x$ and $y_2 \prec x$.

We use $\mathcal{H}^\times$ for the set of $\preceq$-units, and $\mathscr{I}(\mathcal{H})$ for the set of $\preceq$-irreds of the monoid H, which we also refer to as the **irreds** of the premon $\mathcal{H}$.

We will call the premon $H^{\text{div}} := (H, |_H)$ the **divisibility premon** of H.

S. Tringali, An abstract factorization theorem and some applications, *J. Algebra*, 2022.

L. Cossu & S. Tringali, Abstract Factorization Theorems with Applications to Idempotent Factorizations, *Israel J. Math.*, 202?.

Remarks:

1. If H is a Dedekind-finite monoid (i.e., $xy = 1_H$ iff $yx = 1_H$), then $|_H$-unit = unit and $|_H$-irreducible = irreducible. If, in particular, H is commutative and unit-cancellative, then $|_H$-irreducible = atom.

   The classical theory of factorization can be seen as the study of the arithmetic of the divisibility premon of a Dedekind-finite monoid.

2. In some premons $\mathcal{H} = (H, \preceq)$ the elements of $\mathscr{I}(\mathcal{H})$ can be **idempotent** or **invertible** [Tringali 2022, C. & Tringali, 202?].

Preorders vs blow up phenomena

Definition (The shuffling preorder)

Given a premon $\mathcal{H} = (H, \preceq)$, we define the **shuffling preorder** $\sqsubseteq_{\mathcal{H}}$ (read *"squek"*) on $\mathscr{F}(H)$ as follows: for some H-words $\mathfrak{a}$ and $\mathfrak{b}$ with $\|\mathfrak{a}\| = m$ and $\|\mathfrak{b}\| = n$, $\mathfrak{a} \sqsubseteq_{\mathcal{H}} \mathfrak{b}$ if and only if there is an injective function $\sigma\colon [\![1, m]\!] \to [\![1, n]\!]$ such that $\mathfrak{a}[i] \preceq \mathfrak{b}[\sigma(i)] \preceq \mathfrak{a}[i]$ for every $i \in [\![1, m]\!]$.

L. Cossu & S. Tringali, Factorization under Local Finiteness Conditions, *J. Algebra*, 2023.

Remark: The preorder $\sqsubseteq_{\mathcal{H}}$ is artinian[1], therefore every non-empty subset of $\mathscr{F}(H)$ admits a $\sqsubseteq_{\mathcal{H}}$-minimal element.

[1] A preorder $\preceq$ on a set X is artinian if, for every $\preceq$-non-increasing sequence $(x_k)_{k \ge 0}$ of elements of X, there exists $k_0 \in \mathbb{N}$ such that $x_k \preceq x_{k+1}$ for every $k \ge k_0$.

Minimal factorizations

Definition ([Minimal] $\preceq$-factorization)

Let $\mathcal{H} = (H, \preceq)$ be a premon. A **$\preceq$-factorization** of an element $x \in H$ is an $\mathscr{I}(\mathcal{H})$-word $\mathfrak{a} = a_1 * \cdots * a_n$ such that $a_1 \cdots a_n = x$ and we call $\mathcal{Z}_{\mathcal{H}}(x)$ the set of $\preceq$-factorizations of x.

A **minimal $\preceq$-factorization** of x is then a $\sqsubseteq_{\mathcal{H}}$-minimal word in $\mathcal{Z}_{\mathcal{H}}(x)$, i.e., an element $\mathfrak{a} \in \mathcal{Z}_{\mathcal{H}}(x)$ such that there is no $\mathfrak{b} \in \mathcal{Z}_{\mathcal{H}}(x)$ with $\mathfrak{b} \sqsubset_{\mathcal{H}} \mathfrak{a}$. We denote the set of minimal $\preceq$-factorizations of x by $\mathcal{Z}_{\mathcal{H}}^{\mathrm{m}}(x)$.

L. Cossu & S. Tringali, Factorization under Local Finiteness Conditions, *J. Algebra*, 2023.

Remark: If H is commutative and unit-cancellative, then every $|_H$-factorization is a minimal $|_H$-factorization and they are (classical) atomic factorizations.

All the results mentioned from now on come from the preprint

L. Cossu & S. Tringali, On the finiteness of certain factorization invariants, *submitted*, 2023.

Minimal elasticity

We define the **(classical) elasticity** $\varrho(H)$ of a monoid H as the supremum of the set of rational numbers of the form m/n with $m, n \in \mathbb{N}^+$ such that $a_1 \cdots a_m = b_1 \cdots b_n$ for some atoms $a_1, \ldots, a_m, b_1, \ldots, b_n \in H$.

Definition (Minimal elasticity)

Given a premon $\mathcal{H} = (H, \preceq)$,

- The **minimal elasticity** $\varrho^{\mathsf{m}}_{\mathcal{H}}(x)$ of an element $x \in H$ is defined as:

$$\varrho^{\mathsf{m}}_{\mathcal{H}}(x) := \sup \left\{ \|\mathfrak{b}\|^{-1} \|\mathfrak{a}\| : \varepsilon \neq \mathfrak{a}, \mathfrak{b} \in \mathcal{Z}^{\mathsf{m}}_{\mathcal{H}}(x) \right\} \in [0, +\infty].$$

- The **minimal elasticity** $\varrho^{\mathsf{m}}(\mathcal{H})$ of the premon $\mathcal{H}$ is then defined as:

$$\varrho^{\mathsf{m}}(\mathcal{H}) := \sup \{ \varrho^{\mathsf{m}}_{\mathcal{H}}(x) : x \in H \setminus \mathcal{H}^{\times} \} \in [0, +\infty].$$

In particular, we let the minimal elasticity of the monoid H be the minimal elasticity of H^{div}.

Remark: The *minimal elasticity* of a commutative, unit-cancellative monoid is exactly the *classical elasticity*.

A finiteness result for the minimal elasticity

Theorem 1

The minimal elasticity of a *commutative* premonoid with finitely many irreducibles is finite.

Proof: This is a consequence of a more general, purely combinatorial result, using *Dickson's lemma* in a different way than in the classical setting.

Corollary 1

If a commutative monoid is finitely generated modulo units, then its minimal elasticity is finite.

Corollary 2 (cf. Fan *et. al* 2017, Anderson *et. al* 1993)

If a commutative unit-cancellative monoid is finitely generated modulo units, then its elasticity is finite.

Remarks:

- If a commutative unit-cancellative monoid is finitely generated modulo units, then its elasticity is not only finite but also **rational**. The question if, under the hypothesis of Theorem 1, $\varrho^{\mathrm{m}}(\mathcal{H})$ is rational remains **open**.
- No other result comparable to Theorem 1 exists for highly non-cancellative monoids.
- In the above theorem, we cannot get rid of the assumption that H is commutative. We proved that, if $A = \{a, b, c\}$ and $R = \{(\mathfrak{s}_n, \mathfrak{t}_n) \colon n = 2, 3, \ldots\}$ with $\mathfrak{s}_n := c * a^{*n} * b^{*2^n} * a^{*n} * c$ and $\mathfrak{t}_n := a * c^{*n} * b^{*n} * c^{*n} * a$, then $H = Mon\langle A | R\rangle$ is a reduced, atomic, 3-generated, and cancellative monoid with $\varrho^{\mathrm{m}}(H^{\mathrm{div}}) = \infty$. This means that for every $n \in \mathbb{N}^+$ there exist minimal atomic factorizations $\mathfrak{a}_n$ and $\mathfrak{b}_n$ of a non-unit $x_n \in H$ s.t. $\|\mathfrak{a}_n\| \geq n\|\mathfrak{b}_n\|$.

Given a set X and a relation R on the free monoid $\mathscr{F}(X)$, we take $Mon\langle X|R\rangle := \mathscr{F}(X)/\equiv_R$, where $\equiv_R$ is the smallest congruence on $\mathscr{F}(X)$ containing R. $Mon\langle X|R\rangle$ is called a presentation.

Minimal length sets and their unions

The following concepts extend those of *set of lengths*, *system of sets of lengths*, and *union of sets of lengths* from the classical theory.

Definition (Minimal length sets and their unions)

Given a premon $\mathcal{H} = (H, \preceq)$ and an element $x \in H$, we let

$$\mathsf{L}^{\mathsf{m}}_{\mathcal{H}}(x) := \left\{\|\mathfrak{a}\| : \mathfrak{a} \in \mathcal{Z}^{\mathsf{m}}_{\mathcal{H}}(x)\right\} \subseteq \mathbb{N}$$

be the **minimal length set** of x (relative to $\mathcal{H}$). Accordingly, we refer to

$$\mathscr{L}^{\mathsf{m}}(\mathcal{H}) := \left\{\mathsf{L}^{\mathsf{m}}_{\mathcal{H}}(x) \colon x \in H \setminus \mathcal{H}^{\times}\right\}$$

as the **system of minimal length sets** of $\mathcal{H}$; and given $k \in \mathbb{N}$, we call

$$\mathscr{U}^{\mathsf{m}}_{k}(\mathcal{H}) := \bigcup \{L \in \mathscr{L}^{\mathsf{m}}(\mathcal{H}) \colon k \in L\}$$

the **union of minimal length sets** containing k.

Other finiteness results

Proposition 1

The following hold for a premon $\mathcal{H} = (H, \preceq)$:

- $\varrho^{\mathrm{m}}(\mathcal{H}) = 1$ if and only if $|\mathsf{L}^{\mathrm{m}}_{\mathcal{H}}(x)| = 1$ for each $x \in H \setminus \mathcal{H}^\times$, i.e., if and only if $\mathcal{H}$ is **"minimal HF"**.
- If $\varrho^{\mathrm{m}}(\mathcal{H})$ is finite, then $\mathscr{U}^{\mathrm{m}}_k(\mathcal{H})$ is finite for every $k \in \mathbb{N}$.

Theorem 2

Let $\mathcal{H} = (H, \preceq)$ be a premon and suppose there is a finite set $A \subseteq \mathscr{I}(\mathcal{H})$ such that every $\preceq$-irred is $\preceq$-equivalent to an element of A. Then the minimal length sets of $\mathcal{H}$ are all finite, i.e., $\mathcal{H}$ is **"minimal BF"**.

Corollary 3 (cf. Geroldinger and Lettl 1990)

In a premon $\mathcal{H} = (H, \preceq)$ with finitely many $\preceq$-irreds, unions of minimal length sets are all finite.

An easy example

Let H be the multiplicative monoid $\mathbb{Z}/p^n\mathbb{Z}$ of the integers modulo p^n, where $p \in \mathbb{N}$ is a prime and n is an integer ≥ 2.

H is an atomic monoid and the atoms (resp., the units) of H are precisely the $|_H$-irreds (resp., the $|_H$-units).

In addition, every non-zero non-unit of H has an essentially unique atomic factorization, with "essentially unique" meaning that any two atomic factorizations of the same element are equivalent with respect to the shuffling preorder induced by the divisibility preorder $|_H$.

On the other hand, the residue class of 0 modulo p^n has an essentially unique minimal atomic factorization (of length n), but atomic factorizations of any length $\geq n$.

It follows that H is **not BF** but **minimal factorial**: the elasticity $\varrho(H) = \infty$; the minimal elasticity $\varrho^{\mathrm{m}}(H) = 1$; and for every $k \in \mathbb{N}^+$, we have

$$\mathscr{U}_k(H) = \begin{cases} \{k\} & \text{if } 1 \leq k < n, \\ [\![k, \infty]\!] & \text{if } k \geq n \end{cases} \qquad \text{and} \qquad \mathscr{U}_k^{\mathrm{m}}(H) = \begin{cases} \{k\} & \text{if } 1 \leq k \leq n, \\ \emptyset & \text{if } k > n. \end{cases}$$

Here, $\mathscr{U}_k(H)$ denotes the *classical union of length sets* containing k.

THANK YOU