On p-Frobenius of affine semigroups

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Ongoing work with

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Definitions

$S \subset \mathbb{N}^q$ is a set containing 0 and closed under addition

$A = \{a_1, \ldots, a_h\} \subset \mathbb{N}^p$ is a generating set of S if $S = \{\sum_{i=1}^h \lambda_i a_i \mid \lambda_1, \ldots, \lambda_h \in \mathbb{N}\}$

A is a minimal generating set if it is the minimal set, according to inclusion, generating S, denote it by $S = \langle A \rangle$

All our monoids are finitely generated

An affine semigroup is a finitely generated submonoid of $\mathbb{N}^q$

For $q = 1$, S is called a numerical semigroup whenever $S \subseteq \mathbb{N}$ and $\#(\mathbb{N} \setminus S) < \infty$ (equivalently $\gcd(a_1, \ldots, a_h) = 1$)

Definitions

If $n \in S$, then $Z_n(S) = \{\lambda = (\lambda_1, \dots, \lambda_h) \in \mathbb{N}^h \mid n = \sum_{i=1}^h \lambda_i a_i\}$

The minimum integer cone containg S a affine semigroup is

$$\mathcal{C}(S) = \{\sum_{i=1}^{h} \delta_i a_i \mid \delta_i \in \mathbb{Q}_{\geq 0}\}$$

$\mathcal{C}(S)$ has always a finite number of extremal rays (there exist $\{\tau_1, \dots, \tau_r\} \subseteq S$ generating $\mathcal{C}(S)$)

We fix $\preceq$ a monomial order on $\mathbb{N}^q$ (a total order compatible with $+$ in $\mathbb{N}^q$ and such that $0 \preceq x$ for all $x \in \mathbb{N}^q$)

Frobenius elements and vectors

The Frobenius element of S a numerical semigroup is $F(S) = \max(\mathbb{Z} \setminus S)$
If $S = \mathbb{N}$, then $F(S) = -1$

If $S = \langle a, b \rangle$, then $F(S) = ab - a - b$

The Frobenius number f of S is the maximum integer f satisfying that $Z_f(S) = \emptyset$

If $\mathcal{C}(S) \setminus S$ is finite, f the Frobenius vector of S is $\max_{\preceq}(\mathcal{C}(S) \setminus S)$ (see [GMV18])
In this case, f is also the maximum in $\mathcal{C}(S) \setminus S$ with respect to $\preceq$ such that $Z_f(S) = \emptyset$

p-Frobenius vector

The p-Frobenius of S is the element ($p \in \mathbb{N}$)

$$F_p(S) = \max_{\preceq}\{n \in \mathcal{C}(S) \mid 0 < \#Z_n(S) \leq p\}$$

(see [KY23] and [Bro+10])

$$\exists F_p(S) \Leftrightarrow \{n \in \mathcal{C}(S) \mid 0 < \#Z_n(S) \leq p\} \text{ is bounded}$$

Other definition:

$$g_p(S) = \max_{\preceq}\{n \in \mathcal{C}(S) \mid \#Z_n(S) = p\}$$

At least when A is not a m.s.g. $\{g_p(S)\}_{p\in\mathbb{N}}$ is not always an increasing sequence

One of our goals is to provide algorithms for computing p-Frobenius vector in numerical semigroups and $\mathcal{C}$-semigroups

Presentations of semigroups

Every finitely generated commutative monoid is isomorphic to a quotient of the form

$$\mathbb{N}^h/\sigma$$

with σ congruence on $\mathbb{N}^h \times \mathbb{N}^h$, a equivalence relation compatible with the addition (see [RG99])

$$\sum_{i=1}^{h} \alpha_i a_i = \sum_{i=1}^{h} \beta_i a_i \Leftrightarrow [\alpha]_\sigma = [\beta]_\sigma$$

Presentations of semigroups

If we consider the S-graded polynomial ring, the S-homogeneus ideal $I_S \subset \mathbb{K}[x_1, \ldots, x_h]$ is the set

$$\langle \{x_1^{\alpha_1} \ldots x_h^{\alpha_h} - x_1^{\beta_1} \ldots x_h^{\beta_h} \mid (\alpha, \beta) \in \sigma\} \rangle$$

Given a Gröbner basis G of I_S, denote by $\texttt{NormalForm}_{\preceq}(f, G)$ the remainder of the division of $f \in \mathbb{K}[x_1, \ldots, x_h]$ according to $\preceq$

$$\texttt{NormalForm}_{\preceq}(X^{\alpha}, G) = \texttt{NormalForm}_{\preceq}(X^{\beta}, G) \Leftrightarrow [\alpha]_{\sigma} = [\beta]_{\sigma}$$

A k-th elimination order $\preceq_k$ is a monomial order suchth at $x_k \succ x_i$ for every $i \neq k$
This is used to know if a multiple of a generator can be expressed by using the other generators

Main result

Theorem

Let $S = \langle a_1, \ldots, a_h \rangle \subset \mathbb{N}^q$ be an affine semigroup, $p \in \mathbb{N} \setminus \{0\}$, and $\preceq$ a monomial ordering on $\mathbb{N}^q$. Then, there exists $F_p(S)$ if and only if for every $k \in \{1, \ldots, h\}$, there exist $\lambda_k, \alpha_i \in \mathbb{N}$ such that $\lambda_k a_k = \sum_{i=1, i\neq k}^{h} \alpha_i a_i$.

$$\exists F_p(S)$$
$$\Updownarrow$$
$$\forall i \in \{1, \ldots, r\} \exists k_i \mid \#Z_{k_i \tau_i}(S) > 1$$
$$\Updownarrow$$

In every extremal ray there are at least two minimal generators of S

$$\exists F_p(S) \implies \left(\forall p, \forall x \in S,\ 0 < \#Z_x(S) \leq p \implies Z_x(S) \subseteq \left\{ \sum_{i=1}^{h} \beta_i a_i \mid \beta_i \leq p\lambda_i \right\} \right)$$

Computation of $F_p(S)$ and $g_p(S)$

Input: A minimal system of generators $\{a_1, \ldots, a_h\}$ of S and $p \in \mathbb{N} \setminus \{0\}$.
Output: $F_p(S)$ and $g_p(S)$.

$\mathcal{G} \leftarrow$ a generating set of the ideal I_S
$\Lambda = (\lambda_1, \ldots, \lambda_h) \leftarrow (0, \ldots, 0) \in \mathbb{N}^h$
if $p \neq 0$ **then**
  for $k \in [h]$ **do**
    $\mathcal{B} \leftarrow$ (reduced) Gröbner basis of I_S respect a k-th elimination order
    if x_k^α is a monomial of a binomial in $\mathcal{B}$ **then**
      $\lambda_k \leftarrow \alpha$
    end if
  end for
end if
$D \leftarrow \mathcal{D}(\Lambda, p)$
return $F_p(S) = \max_{\preceq}\{n \in D \mid \#\mathrm{Z}_n(S) \leq p\}$ and $g_p(S) = \max_{\preceq}\{n \in D \mid \#\mathrm{Z}_n(S) = p\}$

Optimizations

- D is the bounded set where we search for $F_p(S)$ and $g_p(S)$
- We compute $\{\sum_{i=1}^{h} \gamma_i a_i \mid \gamma \in D\} \subseteq S$
- We sort the above set
- Starting from the maximum and decreasing according to $\preceq$ we check if the number of expressions of the element is equal to p

Improvements of this algorithm are done for $p = 1$ and $p = 2$

Improved computation of $F_1(S)$ and $g_1(S)$

Input: A minimal system of generators $\{a_1, \ldots, a_h\}$ of S.

Output: $F_1(S)$ and $g_1(S)$.

if there is an extremal ray of $\mathcal{C}(S)$ with only one minimal generator of S **then**

  return $\nexists F_1(S)$ and $\nexists g_1(S)$

end if

$\mathcal{B} \leftarrow$ a Gröbner basis of I_S $\Omega \leftarrow \{\alpha \in \mathbb{N}^h \mid \alpha$ is a monomial of a binomial of $\mathcal{B}\}$

$D \leftarrow \{x \in \mathbb{N}^h \mid$ there is no $\alpha \in \Omega$ such that $\alpha \leq x\}$

return $g_1(S) = \max_{\preceq}\{\sum_{i=1}^h \gamma_i a_i \mid (\gamma_1, \ldots, \gamma_h) \in D\}$ and

$F_1(S) = \max_{\preceq}\{F_0(S), g_1(S)\}$

Indispensable binomials

Lemma ([OV10])

Let S such that there is an element $m=\sum_{i=1}^h \alpha_i a_i \in S$ with $\#\mathrm{Z}_{\sum_{i=1}^h \alpha_i a_i}(S)=2$. Then, there is at least an indispensable binomial in I_S.

Corollary

Given S an affine semigroup satisfying the hypothesis of Theorem 1. If there is no indispensable binomial in I_S, then $g_2(S)=\emptyset$

Improved computation of $g_2(S)$

Input: A minimal system of generators $\{a_1,\dots,a_h\}$ of S.
Output: $F_2(S)$ and $g_2(S)$.

$\mathcal{G} \leftarrow$ a generating set of the ideal I_S
$\Lambda = (\lambda_1,\dots,\lambda_h) \leftarrow (0,\dots,0) \in \mathbb{N}^h$
for $k \in [1,h]$ **do**
  $\mathcal{B} \leftarrow$ (reduced) Gröbner basis of I_S respect a k-th elimination order $\preceq_k$
  if x_k^α is a monomial of a binomial in $\mathcal{B}$ **then**
    $\lambda_k \leftarrow \alpha$
  end if
end for
$D \leftarrow \{\gamma = (\gamma_1,\dots,\gamma_h) \in \mathcal{D}'(\Lambda) \mid \mathtt{NormalForm}_{\preceq_k}(X^\gamma,\mathcal{B}) = X^\gamma\}$
$D \leftarrow \{\gamma = (\gamma_1,\dots,\gamma_h) \in D \mid X^\gamma \notin I_v\}$
$g \leftarrow \max_{\preceq}\{\sum_{i=1}^h \gamma_i a_i \mid (\gamma_1,\dots,\gamma_h) \in D\}$
$f \leftarrow \max_{\preceq}\{F_0(S), g\}$
$D \leftarrow D \setminus \mathcal{D}'(\Lambda)$
if There is no indispensable binomial in I_S **then return** $g_2(S) = \emptyset$ and $F_2(S) = f$
end if
$I \leftarrow$ the set of indispensable binomials in I_S
$G \leftarrow \emptyset$
while $D \neq \emptyset$ **do**
  if there is $\gamma,\gamma' \in D$ with $X^\gamma - X^{\gamma'} = bX^\delta$ with $b \in I$ **then**
    $G \leftarrow G \cup \{\{\gamma,\gamma'\}\}$
  else
    $D \leftarrow D \setminus \{\gamma,\gamma'\}$
  end if
end while
  return $g_2(S) = \max_{\preceq}\{\sum_{i=1}^h \gamma_i a_i \mid (\gamma_1,\dots,\gamma_h) \in G$ and $\#Z_{\sum_{i=1}^h \gamma_i a_i}(S) = 2\}$ and $F_2(S) = \max_{\preceq}\{f, g_2(S)\}$

Improved computation of $g_2(S)$

We look for the elements of $g_2(S)$ in the set

$$D \cap (\cup_{\lambda \in \Omega}(\lambda + \mathbb{N}^p))$$

with Ω the set of exponents of the indispensable binomials and D the same set of the first algorithm

Gluing semigroups

Gluing of semigroups arises from the study of complete intersetion numerical semigroups (see [Del76] and [SL22, Chapter 8])

$S = \langle a_1, \ldots, a_h \rangle \subset \mathbb{N}$, $d \in \mathbb{N}$ and $\gamma \in S \setminus \{a_1, \ldots, a_h\}$ with $\gcd(d, \gamma) = 1$
$S \oplus_{d,\gamma} \mathbb{N}$ is the affine semigroup minimally generated by $\{da_1, \ldots, da_h, \gamma\}$

We say that $S \oplus_{d,\gamma} \mathbb{N}$ is a $\mathbb{N}$-gluing

Numerical semigroups of the form $S \oplus_{d,\gamma} \mathbb{N}$ ($\mathbb{N}$-gluing) fulfill that

$$F(S \oplus_{d,\gamma} \mathbb{N}) = dF(S) + (d-1)\gamma$$

p-Frobenius and $\mathbb{N}$-gluings

Lemma

Let $s' = ds + a\gamma \in S \oplus_{d,\gamma} \mathbb{N}$ with $s \in S$ and $0 \le a \le d-2$. Then, $\#Z_{s'}(S \oplus_{d,\gamma} \mathbb{N}) = \#Z_{s'+\gamma}(S')$.

$$\#Z_{ds}(S \oplus_{d,\gamma} \mathbb{N}) = \#Z_{ds+\gamma}(S \oplus_{d,\gamma} \mathbb{N}) = \cdots = \#Z_{ds+(d-1)\gamma}(S \oplus_{d,\gamma} \mathbb{N})$$

Lemma

$F_p(S \oplus_{d,\gamma} \mathbb{N}) \le dF_p(S) + (d-1)\gamma$.

Theorem

Assume that $\#\mathrm{Z}_{F_p(S)}(S) = p$. Then, $F_p(S \oplus_{d,\gamma} \mathbb{N}) = dF_p(S) + (d-1)\gamma$ if and only if for every $b \in \mathrm{Z}_\gamma(S)$ there is no $a \in \mathrm{Z}_{F_p(S)}(S)$ such that $b \le_{\mathbb{N}^h} a$.

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