# Unique Maximal Rings of Functions

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- While $M_0(G)$ is a simple near-ring, it does contain rings of functions.
- For example, if $G$ is abelian, $\mathrm{End}(G)$, under the same operations, is a ring contained in $M_0(G)$.

# Rings determined by Covers of Groups

- Let $C := \{A_\alpha \mid \alpha \in \mathcal{A}\}$ be an abelian cover of $G$, i.e., each cell $A_\alpha$ is an abelian subgroup of $G$ and $\cup_{\alpha \in \mathcal{A}} A_\alpha = G$.

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- $C$ determines a ring $\mathcal{R}(C)$, of zero preserving functions on $G$, defined by $\mathcal{R}(C) := \{f \in M_0(G) \mid f|_{A_\alpha} \in \mathrm{End}(A_\alpha)$ for all $\alpha \in \mathcal{A}\}$. We call $\mathcal{R}(C)$ *the ring determined by the cover $C$.* Note that the zero function, 0, and the identity function, id, are in $\mathcal{R}(C)$.

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- $C$ determines a ring $\mathcal{R}(C)$, of zero preserving functions on $G$, defined by $\mathcal{R}(C) := \{f \in M_0(G) \mid f|_{A_\alpha} \in \mathrm{End}(A_\alpha) \text{ for all } \alpha \in \mathcal{A}\}$. We call $\mathcal{R}(C)$ *the ring determined by the cover $C$*. Note that the zero function, 0, and the identity function, id, are in $\mathcal{R}(C)$.
- On the other hand, let $S$ be a ring in $M_0(G)$. Then $\mathcal{C}(S) := \{B \subseteq G \mid B \text{ is an abelian subgroup of } G \text{ and } S|_B \subseteq \mathrm{End}(B)\}$ is an abelian cover of $G$, called the *cover of $G$ determined by the ring $S$*.

# A Galois Connection

- **Theorem 1.1 (Cannon, Maxson, Neuerburg, 2008)** *Let $G$ be a group, let $\Gamma$ denote the collection of abelian covers of $G$ and let $\Lambda$ denote the collection of rings in $M_0(G)$. Then the maps* $\mathcal{R} : \Gamma \to \Lambda,\ C \mapsto \mathcal{R}(C)$ *and* $\mathcal{C} : \Lambda \to \Gamma,\ S \mapsto \mathcal{C}(S)$*, determine a Galois connection between* $\Gamma$ *and* $\Lambda$*.*

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- For any abelian cover $C$, $\mathcal{C}\mathcal{R}(C) \supseteq C$. Moreover, $\mathcal{R}\mathcal{C}\mathcal{R}(C) = \mathcal{R}(C)$. We call $\mathcal{C}\mathcal{R}(C)$ the closure of $C$ and denote this by $\overline{C}$. The cover $C$ is *closed* if $C = \overline{C}$.

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- Also, for any ring $T$ in $M_0(G)$, $T \subseteq \mathcal{R}\mathcal{C}(T)$, so when $T$ is a maximal ring, $T = \mathcal{R}\mathcal{C}(T)$. Hence $T$ is determined by some abelian cover of $G$.

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- For any abelian cover $C$, $\mathcal{CR}(C) \supseteq C$. Moreover, $\mathcal{RCR}(C) = \mathcal{R}(C)$. We call $\mathcal{CR}(C)$ the closure of $C$ and denote this by $\overline{C}$. The cover $C$ is *closed* if $C = \overline{C}$.

- Also, for any ring $T$ in $M_0(G)$, $T \subseteq \mathcal{RC}(T)$, so when $T$ is a maximal ring, $T = \mathcal{RC}(T)$. Hence $T$ is determined by some abelian cover of $G$.

- When $M_0(G)$ contains a unique maximal ring, we say $G \in \mathcal{UMR}$.

# Some Basic Results

- **Theorem 1.2 (Kreuzer, Maxson, 2006)** *Let $A$ be an abelian group. If $A$ is a torsion group or finitely generated, then $\mathrm{End}(A)$ is a maximal ring in $M_0(A)$.*

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- **Corollary 1.4** *Let $G$ be a finite group. If there exists an abelian cover $D$ of $G$ such that* $\mathcal{R}(D) \nsubseteq \mathcal{R}(M_c)$ *then* $G \notin \mathcal{UMR}$.

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- **Corollary 1.4** *Let $G$ be a finite group. If there exists an abelian cover $D$ of $G$ such that* $\mathcal{R}(D) \not\subseteq \mathcal{R}(M_c)$ *then* $G \notin \mathcal{UMR}$.
- **Corollary 1.5** *If $G$ is a finite group and every maximal cyclic subgroup is also maximal as an abelian subgroup, then* $G \in \mathcal{UMR}$.

# Abelian Groups

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- **Lemma 2.3** *If $A$ is a torsion abelian group, $A = \oplus_p A_p$, such that each $A_p$ is a bounded group. Then $A \in \mathcal{UMR}$ if and only if each $A_p$ is cyclic. In this case, $\mathrm{End}(A)$ is the unique maximal ring in $M_0(A)$.*

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- **Theorem 2.4** *Let $A$ be a finitely generated abelian group. Then $A \in \mathcal{UMR}$ if and only if $A$ is cyclic.*

# Finite Nilpotent Groups

- If $G$ is finite and nilpotent, then $G = S(p_1) \oplus \cdots \oplus S(p_t)$, the decomposition of $G$ into the direct sum of its Sylow subgroups $S(p_i),\ i = 1, \ldots, t$. It is known that if $R$ is a maximal ring in $M_0(G)$, then $R \cong R_1 \oplus \cdots \oplus R_t$ where $R_i$ is a maximal ring in $M_0(S(p_i))$ for each $i = 1, \ldots, t$.

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- **Theorem 3.1** *Let $G$ be a finite p-group. Then $G \in \mathcal{UMR}$ if and only if $p = 2$ and $G$ is cyclic or a generalized quaternion group, or $p \geq 3$ and $G$ is cyclic.*

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- **Corollary 3.2** *Let $G$ be a finite nilpotent group. Then $G \in \mathcal{UMR}$ if and only if its 2-Sylow subgroup is cyclic or a generalized quaternion group, and its $p$-Sylow subgroups for odd $p$ are cyclic.*

# The symmetric groups $S_n$

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- **Theorem 4.1** *Let* $\sigma = t_1[k_1] + t_2[k_2] + \cdots + t_r[k_r] \in S_n$, *where the* $k_i$ *are all different and the integers* $t_i \geq 1$ *for all* $i = 1, \ldots, r$. *Then* $\langle \sigma \rangle$ *is not maximal cyclic in* $S_n$ *if and only if there exist partitions* $t_i = s_{i,1} + \cdots + s_{i,y_i}$ *for each* $i$ *(where the* $s_{i,j}$ *are positive integers), with at least one* $s_{i,j} \geq 2$, *and an integer* $q$ *such that* $s_{i,j} | q$ *and* $\gcd\left(\frac{q}{s_{i,j}}, k_i\right) = 1$ *for all* $i$ *and* $j$.

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- Example: In $S_{12}$, $\langle\sigma\rangle = \langle[2] + [2] + [4] + [4]\rangle$ is not maximal cyclic.
  In $S_{16}$, $\langle\sigma\rangle = \langle[3] + [3] + [4] + [6]\rangle$ is maximal cyclic.
  In $S_n$, an $n - 4$ cycle generates a maximal cyclic subgroup if and only if $n \equiv 4 \pmod 6$.

# The symmetric groups $S_n$

- Let $\mathcal{P}$ be a partition of $M = \{1, 2, \ldots, n\}$. For $K \in \mathcal{P}$, define $+_K$ such that $(K, +_K)$ is an abelian group. Consider the sequence $a = (a_K)_{K \in \mathcal{P}},\ a_K \in K$. Define $f_a : M \to M$ by $f_a(b) = a_K +_K b$, $(b \in K)$. Then $H = \{f_a\}$ is an abelian subgroup of $S_n$.

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- **Theorem 4.2 (Winkler, 1993)** *$H$ is a maximal abelian subgroup of $S_n$ if and only if $\mathcal{P}$ contains at most one singleton.*

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- **Theorem 4.2 (Winkler, 1993)** *$H$ is a maximal abelian subgroup of $S_n$ if and only if $\mathcal{P}$ contains at most one singleton.*
- **Theorem 4.3** *$S_n \in \mathcal{UMR}$ if and only if $n \in \{3, 5, 7, 9\}$.*

# An Application

- **Theorem 5.1** *Let $G$ be a finite non-abelian group, a finitely generated abelian group, or a torsion abelian group with bounded p-components. Then every subring of $M_0(G)$ is commutative if and only if $G \in \mathcal{UMR}$.*

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- **Corollary 5.2** *For a finite group* $G$*, every subring of* $M_0(G)$ *is commutative if and only if* $G \in \mathcal{UMR}$.