Regular t-ideals of Polynomial Rings and Semigroup Rings with Zero Divisors

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Introduction and Motivation

This talk is based on my paper "Regular t-ideals of polynomial rings and semigroup rings," currently in preparation.

Theorem (D.D. Anderson et al., 1995; cf. Querré, 1980)

If D is integrally closed with quotient field K, then each t-ideal of $D[X]$ has the form $hI[X]$ with $h \in K[X]$ and I a t-ideal of D.

Corollary (Folklore)

A domain D is a UFD, Krull domain, π-domain, (generalized) GCD domain, or PVMD if and only if the same holds for $D[X]$.

Question (Anderson et al., 1985; Glaz, 2000; Lucas, 2005)

When do analogous properties ascend to polynomial/semigroup rings with zero divisors? Are there forms of the above theorem that hold for polynomial/semigroup rings with zero divisors?

A Tale of Two t-operations

Throughout, let R be a commutative ring.

Definition (Folklore; Lucas 1989-2005)

1. $\mathrm{Reg}(R) := \{r \in R \mid ((0) :_R (r)) = (0)\}$. An ideal I is **regular** if $I \cap \mathrm{Reg}(R) \neq \emptyset$ and **semiregular** if it has a f.g. faithful subideal.
2. $T(R) := \{a/r \mid a \in R,\ r \in \mathrm{Reg}(R)\}$ is R's **total quotient ring**.
3. $Q_0(R) := \bigcup\{(R :_{T(R[X])} I) \mid I \text{ is a semiregular ideal of } R\}$ is the **ring of finite fractions** of R [Lucas]. Note: $T(R) \subseteq Q_0(R)$, with equality if R has **Property A** (i.e., semiregular ideals are regular).
4. Set $I^{-1} := (R :_{T(R)} I), I^t := \bigcup\{(J^{-1})^{-1} \mid J \text{ is a f.g. } R\text{-submodule of } I\}$ for $I \in \mathrm{Mod}_R(T(R))$. Similarly define $I_0^{-1_0}, I_0^{t_0}$ for $I_0 \in \mathrm{Mod}_R(Q_0(R))$.
5. I is **fractional** if I^{-1} is regular; I_0 is Q_0-**fractional** if $I_0^{-1_0}$ is semiregular.

Note: $R[\{X_\lambda\}_\lambda]$ has Property A and is **Marot** (i.e., regular ideals are regularly generated) and $Q_0(R)[\{X_\lambda\}_\lambda]$ is an overring of $R[\{X_\lambda\}_\lambda]$.

Proposition (Juett, 2023; cf. Lucas, 2005)

$I_0[\{X_\lambda\}_\lambda]$ is a regular fractional ideal of $R[\{X_\lambda\}_\lambda]$ if and only if I_0 is a semiregular Q_0-fractional ideal of R, in which case $I_0[\{X_\lambda\}_\lambda]^t = I_0^{t_0}[\{X_\lambda\}_\lambda]$.

Regular t-ideals and Divisibility Properties

1. A (fractional) ideal I of R is a **(fractional) t-ideal** if $I = I^t$, **t-finite** if $I^t = J^t$ for some finitely generated (fractional) ideal J, **invertible** if $II^{-1} = R$, and **t-invertible** if $(II^{-1})^t = R$.
2. R is **factorial** if every regular nonunit is a unique up to order and associates product of irreducibles [D.D. Anderson & Markanda, 1985]. A Marot ring is factorial if and only if every regular t-ideal is principal.
3. R is a **regular π-ring** if regular proper principal ideals are products of prime ideals, or equivalently regular t-ideals are invertible [Kang, 1991].
4. R is a **GCD ring** if every pair of regular elements has a GCD [D.D. Anderson & Markanda, 1985]. A Marot ring is a GCD ring if and only if every t-finite regular t-ideal is principal [Elliott, 2019].
5. R is a **G-GCD ring** if every pair of invertible ideals has a GCD [Juett, 2023; cf. D.D. & D.F. Anderson, 1980]. A Marot ring is a G-GCD ring if and only if every t-finite regular t-ideal is invertible.
6. A **Glaz (G-)GCD ring** is a (G-)GCD p.p. ring [Glaz, 2000].
7. R is **Krull** if every regular t-ideal is t-invertible [Elliott, 2019].
8. R is a **Prüfer v-multiplication ring** (PVMR) if every t-finite regular t-ideal is t-invertible.

Regular t-ideals of Polynomial Rings

Theorem (Juett, 2023)

1. *Every regular t-ideal of* $R[\{X_\lambda\}_\lambda]$ *has the form* $hI[\{X_\lambda\}_\lambda]$ *with* $h \in T(R)[\{X_\lambda\}_\lambda]$ *and* I *a (regular t-)ideal of* R *if and only if* R *is a finite direct product of integrally closed domains.*
2. *Every t-finite regular t-ideal of* $R[\{X_\lambda\}_\lambda]$ *has the form* $hI[\{X_\lambda\}_\lambda]$ *with* $h \in T(R)[\{X_\lambda\}_\lambda]$ *and* I *a (t-finite regular t-)ideal of* R *if and only if* R *is integrally closed and* $T(R)$ *is von Neumann regular.*

Divisibility Properties of Polynomial Rings

Corollary

1. *$R[\{X_\lambda\}_\lambda]$ is Krull (resp., a regular π-ring, factorial) if and only if R is a finite direct product of Krull domains (resp., π-domains, UFDs) [D.D. Anderson et al., 1985].*
2. *$R[\{X_\lambda\}_\lambda]$ is a PVMR if and only if R is a PVMR and $T(R)$ is von Neumann regular [Juett, 2023; cf. Lucas 2005].*
3. *$R[\{X_\lambda\}_\lambda]$ is a (G-)GCD ring if and only if R is a (G-)GCD ring and $T(R)$ is von Neumann regular [Juett, 2023].*
4. *$R[\{X_\lambda\}_\lambda]$ is a Glaz (G-)GCD ring if and only if R is a Glaz (G-)GCD ring [Juett, 2023].*

Basic Properties of $R[S]$

Throughout, let $(S, +)$ be a nontrivial torsion-free cancellative monoid with difference group G.

Proposition (Folklore)

1. $\mathrm{Reg}(R[S]) = \{f \in R[S] \mid af \neq 0$ *for all* $0 \neq a \in R\}$, *so* $Q_0(R)[S]$ *is an overring of* $R[S]$.
2. $R[S]$ *is Marot with Property A.*
3. *If* $R = \prod_{i=1}^{n} R_i$, *then* $T(R) = \prod_{i=1}^{n} T(R_i)$, $Q_0(R) = \prod_{i=1}^{n} Q_0(R_i)$, *and* $R[S] = \prod_{i=1}^{n} R_i[S]$.

Proposition (Juett, 2023)

$I_0[J]$ *is a regular fractional ideal of* $R[S]$ *if and only if* I_0 *is a semiregular* Q_0*-fractional ideal of* R *and* J *is a nonempty fractional ideal of* S, *in which case* $I_0[J]^t = I_0^{t_0}[J^t]$.

Regular t-ideals of Semigroup Rings

Theorem (Juett, 2023; cf. Chang, 2011)

The following are equivalent.

1. *Every regular t-ideal A of $R[S]$ has the form $A = \prod_{i=1}^{n} h_i I_i[J_i]$, where $R = \prod_{i=1}^{n} R_i$, each $h_i \in T(R_i)[G]$, each I_i is a regular t-ideal of R_i, each J_i is a nonempty t-ideal of S, and we can take each $h_i = 1$ if the monomials of A generate a regular ideal.*
2. *R is a finite direct product of integrally closed domains, S is root closed, and G satisfies the ascending chain condition on cyclic subgroups.*
3. *$R[S]$ is i.c. and $T(R)[G]$ is Krull (or equivalently factorial).*

Theorem (Juett, 2023)

The following are equivalent.

1. *Every t-finite regular t-ideal A of $R[S]$ ...*
2. *R is integrally closed, $T(R)$ is VNR, and S is root closed.*
3. *$R[S]$ is i.c. and $T(R)[G]$ is a PVMR (or equiv. a Glaz GCD ring).*

Divisibility Properties of Semigroup Rings

Corollary (Juett et al., 2021-2023)

1. *$R[S]$ is Krull if and only if R is a finite direct product of Krull domains, S is Krull, and G satisfies the ascending chain condition on cyclic subgroups.*
2. *$R[S]$ is a regular π-ring (resp., factorial) if and only if R is a finite direct product of π-domains (resp., UFDs), S is factorial, and G satisfies the ACC on cyclic subgroups.*
3. *$R[S]$ is a PVMR if and only if R is a PVMR, $T(R)$ is von Neumann regular, and S is a PVMS.*
4. *$R[S]$ is a (G-)GCD ring if and only if R is a (G-)GCD ring, $T(R)$ is von Neumann regular, and S is a GCD monoid.*
5. *$R[S]$ is a Glaz (G-)GCD ring if and only if R is a Glaz (G-)GCD ring and S is a GCD monoid.*

Conclusion and Summary

- I extended the classic result about t-ideals of polynomial domains to polynomial/semigroup rings with zero divisors.
- Application: I determined when polynomial/semigroup rings with zero divisors satisfy several different divisibility properties.
- Thank you for your attention. **Any questions?**