Half-factoriality and length factoriality in monoids and domains

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In The Beginning: a problem of Narkiewicz

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A. Zaks (1976) seems to have been the first one to use the terminology half-factorial domain (HFD). Here is (was) his definition:

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R is an HFD if given any two irreducible factorizations in R:

$$\alpha_1\alpha_2\cdots\alpha_n = \beta_1\beta_2\cdots\beta_m$$

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We note that although the original definition of HFD did not assume that the domain is to be atomic, this is now industry standard. That is to say "atomic" is implicit in the definition of HFD.

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Theorem

The following two statements are not provable in ZF:

1. *Every PID is a UFD.*
2. *Every PID has a maximal ideal.*

Because of this ideal-theoretic characterization of UFDs, there are a number of classical theorem that "fall out" more easily. For example, the facts that if R is a UFD, then so are $R[x]$ and R_S follow "easily" (at least more easily than the proofs presented in many beginning books on the subject). This characterization also gives the equivalence of "PID" and "UFD of dimension no more than 1" almost immediately.

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For the class of HFDs there is no known ideal theoretic characterization (and it is interesting to note that, in general, neither $R[x]$ nor R_S need be HFDs when R is an HFD).

On the other hand, if we again restrict to the case of rings of algebraic integers, we now have Carlitz' theorem to provide us with our needed ideal-theoretic characterization (note that in this case, if R is an HFD, then so are $R[x]$ and R_S).

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Factorization notions tend to be element-wise or defined on the set of principal ideals, but can be a wide gap in these concepts (many wildly non-Noetherian rings satisfy ACCP, for example) and so, finding ideal-theoretic characterizations for properties of this type is something of a holy grail.

Of the classical "factorization types" defined in the "Factorization in Integral Domains" paper (D.D. Anderson, D.F. Anderson, and M. Zafrullah, JPAA 1990), one can argue that the HFD property is the closest to the unique factorization property. And yet, it is one of the most badly behaved with respect to stabilization properties. HFDs are not necessarily preserved in polynomial extensions (only atomicity is also potentially unstable with respect to polynomial adjunction), or localizations (Chapman and Smith even demonstrated this delinquent behavior in Dedekind domains with finite class group).

HFDs do not have to be (completely) integrally closed and so there are more possibilities for them. For example, any construction of the form $F + xK[x]$ or $F + xK[[x]]$ where $F \subseteq K$ is a field is an HFD (and it is a nontrivial example if the containment $F \subset K$ is strict).

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Another good possible source (this with a number-theoretic flavor) are subrings of rings of algebraic integers. In fact, we will narrow further by only considering the orders in the field $\mathbb{Q}(\sqrt{d})$ where $d \in \mathbb{Z}$ is square-free.

That is for every $n \in \mathbb{N}$, we consider the rings $\mathbb{Z}[n\omega]$ where

$$\omega = \left\{ \sqrt{d}, \text{ if } d \equiv 1 \bmod (4), \frac{1+\sqrt{d}}{2}, \text{ if } d \equiv 2, 3 \bmod (4). \right.$$

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One of these rings is integrally closed if and only if $n = 1$ and so, the best that one can do, in a certain sense is the HFD property. Although it "appears" that if $d > 0$ and $\mathbb{Z}[\omega]$ is a UFD, then for infinitely many values of n, $\mathbb{Z}[n\omega]$ is an HFD, this is not true in the imaginary case.

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Theorem

If $d < 0$, and $n > 1$ the ring $\mathbb{Z}[n\omega]$ is an HFD if and only if $d = -3$ and $n = 2$. The ring $\mathbb{Z}[\sqrt{-3}]$ is the unique non-integrally closed imaginary quadratic HFD.

Polynomials

$\mathbb{Z}[\sqrt{-3}]$ is arguably the most interesting/important non-trivial HFD. Note that in the polynomial ring $\mathbb{Z}[\sqrt{-3}][x]$, we have the following factorization of the polynomial $4x^2 + 4x + 4$:

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From this, we obtain the following.

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Surprisingly, the necessary condition "R is integrally closed" required for $R[x]$ to be an HFD, is NOT required for $R[[x]]$ to be an HFD. For example:

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As it turns out the weirdo ring $\mathbb{Z}[\sqrt{-3}][[x]]$ actually is an HFD (even though $\mathbb{Z}[\sqrt{-3}][x]$) is not an HFD. We see rather dramatically that "integrally closed" is not needed for a power series ring to be an HFD.

The Boundary Map

We recall that an integral domain is an HFD if and only if it admits a length function $L : R^* \longrightarrow \mathbb{N}_0$ such that $L(xy) = L(x) + L(y)$, $L(u) = 0$ if and only if u is a unit of R and $L(x) = 1$ if and only if x is irreducible in R.
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As it turns out, the boundary map is useful in studying overrings of HFDs (especially those with nice integrality properties).

Theorem

If R is an HFD and T is an almost integral overring of R, then $\partial_R(t) \geq 0$ for all nonzero $t \in T$ (boundary nonnegative).

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A central question concerning HFDs is when is an overring of an HFD again an HFD. Since this is almost hopeless (even UFDs behave badly) we might restrict to the case of (almost) integral extensions. Here are a couple of known results.

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If R is an HFD order in a ring of algebraic integers, then $\overline{R}$ is also an HFD.

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There exist integral overrings that have boundary 0 nonunits and the extension $\mathbb{Z} + 2x\mathbb{Z}[x] \subset \mathbb{Z}[x]$ shows that "almost" integral is not good enough ($x = \frac{2x}{2}$ is of boundary 0 and is not a unit).

Length Factoriality

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Theorem (AB-WS 2011)

If R is an atomic length factorial domain, then R is a UFD.

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It is an easy exercise to show that every nonunique (and nonredundant) factorization in this monoid is a superposition of this one "master factorization" and put simply, this is the machine that makes the previous theorem work. This nontrivial factorization works well in monoids, but is not very compatible with two binary operations with the distributive property.

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One should also notice that in the above factorization, 2 is a "long" irreducible and 3 is a "short" irreducible. We make this more precise:

Definition

Let M be a commutative cancellative atomic monoid and $x \in M$ an irreducible. We say x is "purely short" (resp. long) if whenever we have irreducible factorizations

$$x\alpha_1\alpha_2\cdots\alpha_m = \beta_1\beta_2\cdots\beta_n,$$

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This notion leads us to an interesting result.

Theorem (SC-AB-FG-WS, 2021)

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In any atomic monoid, the set of long irreducibles and the set of short irreducibles are both finite sets. Additionally, in an integral domain, at least one of these sets must be empty.

So in a domain, if there is a long, there are no shorts and vice versa. One can construct Dedekind domains to show that there can exist longs and shorts in domains.

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Theorem (AB-BM, 2011)

Let M be any reduced, cancellative, torsion-free monoid. Then there exists an integral domain with atomic factorization structure isomorphic to M.

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Question

Classify the (reduced, atomic) monoids that may be realized as the (reduced) monoid of an **atomic** *integral domain.*

To finish up, here are a couple of recent advances.

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Theorem (JB-AB-CM preprint 2023)

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Theorem (Squeeze Theorem JB-AB-CM preprint 2023)

If R is an order in a ring of algebraic integers with integral closure $\overline{R}$ and T is such that

$$R \subseteq T \subseteq \overline{R}$$

then T is an HFD.

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5. *Is there an elasticity squeeze theorem?*

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We note that the radical conductor condition applies to all quadratic orders that are HFDs. Note that this is a large contrast with the polynomial situation.

Thanks for listening! And much thanks and appreciation to the organizers!