The $D + M$ Construction in Semidomains

Harold Polo

University of California, Irvine

(joint work with F. Gotti)

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Background: Monoids

Throughout this talk, a *monoid* is a semigroup with identity that is cancellative and commutative.

We denote by $\mathcal{U}(M)$ the set of invertible elements of M, and M is *reduced* if $\mathcal{U}(M) = \{0\}$.

Given a monoid M, we denote by $\mathcal{G}(M)$ the *group of differences* of M.

Background: Semirings

A *commutative semiring* S is a nonempty set endowed with two binary operations denoted by '$+$' and '$\cdot$' and called *addition* and *multiplication*, respectively, such that the following conditions hold:

- $(S, +)$ is a monoid with its identity element denoted by 0;
- $(S, \cdot)$ is a commutative semigroup with an identity element denoted by 1;
- $b \cdot (c + d) = b \cdot c + b \cdot d$ for all $b, c, d \in S$;
- $0 \cdot b = 0$ for all $b \in S$.

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A subset S' of a semiring S is a *subsemiring* of S if $(S', +)$ is a submonoid of $(S, +)$ that contains 1 and is closed under multiplication.

Background: Semirings

If R and S are semirings, then a function $\varphi\colon R \to S$ is a *semiring homomorphism* if the following conditions hold:

1. $\varphi(0_R) = 0_S$;
2. $\varphi(1_R) = 1_S$;
3. $\varphi(x + y) = \varphi(x) + \varphi(y)$;
4. $\varphi(x \cdot y) = \varphi(x) \cdot \varphi(y)$.

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Sometimes we will abuse notation and say that $s_1 - s_2 \in S$ provided that there exists $s \in S$ such that $s + s_2 = s_1$.

A semiring S is *yoked* provided that, for every $s_1, s_2 \in S$, either $s_1 - s_2 \in S$ or $s_2 - s_1 \in S$. A *semifield* is a semiring in which every nonzero element has a multiplicative inverse.

Background: Ideals

An *ideal* I of a semiring S is a nonempty subset of S satisfying the following two conditions:

1. if $x, y \in I$, then $x + y \in I$;
2. if $s \in S$ and $x \in I$, then $sx \in I$.

An ideal I is *maximal* provided that $I \neq S$ and, for any ideal J containing I, we have that either $J = I$ or $J = S$. An ideal I is *prime* provided that $I \neq S$ and if $a \cdot b \in I$, then either $a \in I$ or $b \in I$.

Ex: Observe that all multiples of $k \in \mathbb{N}_{>1}$ form an ideal of the semiring $\mathbb{N}_0$. Also note that $\mathbb{N}_0 \setminus \{1\}$ is the only maximal ideal of $\mathbb{N}_0$.

Background: Ideals

An ideal I of a semiring S is called *subtractive* if $s + a \in I$ for $s \in S$ and $a \in I$, then $s \in I$. We denote by $\mathcal{I}(S)$ the set consisting of all subtractive ideals of the semiring S.

A maximal element of $\mathcal{I}(S)$ (with respect to inclusion) is called a *subtractive maximal ideal* of S.

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Proposition (Golan, 1999)

An ideal I of a semiring S is the kernel of a semiring homomorphism if and only if I is subtractive.

Background: Semimodules

Let R be a (commutative) semiring. An *R-semimodule* consists of an additively written commutative monoid M and a map from $R \times M$ to M satisfying the following:

1. $r(m + n) = rm + rn$;
2. $(r + s)m = rm + sm$;
3. $(rs)m = r(sm)$;
4. $1m = m$;
5. $0m = 0$.

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Lemma (Gotti and P., 2023)

For a semiring S, the following conditions are equivalent.

(a) S is a semidomain.

(b) The multiplication of S extends to $\mathcal{G}(S)$ turning $\mathcal{G}(S)$ into an integral domain.

Subtractive retracts

When can we represent a semidomain T as $K + M$, where K is a yoked semifield and M is a subtractive maximal ideal of T?

Recall: A semiring S is *yoked* provided that, for every $s_1, s_2 \in S$, either $s_1 - s_2 \in S$ or $s_2 - s_1 \in S$. A *semifield* is a semiring in which every nonzero element has a multiplicative inverse.

Subtractive retracts

Definition

Let T be a semiring, and let S be a subsemiring of T. We say that S is a *retract* of T provided that there exists a semiring homomorphism $\varphi\colon T \to S$ called a *retraction* that is the identity mapping on S.

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Ex: Consider the semidomain

$$S = \{c_0 + c_1x + \cdots + c_nx^n \in \mathbb{N}_0[x] \mid \text{ either } c_0 \neq 0 \text{ or } c_0 = c_1 = 0\}.$$

Note that the semiring homomorphism $\varphi\colon S \to \mathbb{N}_0$ given by $\varphi(p) = p(0)$ is a retraction. However, $\mathbb{N}_0$ is not a direct summand of S as an $\mathbb{N}_0$-semimodule.

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Definition

Let $\varphi\colon T \to S$ be a semiring retraction. If $x - \varphi(x) \in T$ for all $x \in T$, then we say that the semidomain S is a *subtractive retract* of T.

The $K + M$ representation

Theorem (Gotti and P., 202?)

Let T be a semidomain, and let K be a yoked subsemiring of T. Then K is a semifield that is a subtractive retract of T if and only if there exists a subtractive maximal ideal M of T such that $T = K + M$.

Ex: $\mathbb{R}_{\geq 0} + x\mathbb{R}[x]$; $\mathbb{R}_{\geq 0} + x\mathbb{R}_{\geq 0}[x]$; $\mathbb{R}_{\geq 0} + x\mathbb{R}[\![x]\!]$

Remark: If K is a yoked semifield that is a subtractive retract of a semidomain T, then $\mathcal{G}(K)$ is a field that is a retract of the integral domain $\mathcal{G}(T)$.

The $K + M$ Representation

Proposition (Gotti and P., 202?)

An additively reduced semidomain T has at most one representation of the form $K + M$ and if $T = K + M$, then $K = \{0\} \cup T^{\times}$.

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Example

Let $T = K[x]$, where K is an ordered field. So $T = K + xK[x]$ and $T = K + M$ with $M = \{\sum_{i=0}^{n} d_i x^i \mid \sum_{i=0}^{n} d_i = 0\}$. Observe that $M \neq xK[x]$ since $x - 1 \in M$. Consequently, the integral domain $K[x]$ admits at least two representations of the form $K + M$ for an arbitrary ordered field K. However, the semidomain $K_{\geq 0}[x] = K_{\geq 0} + xK_{\geq 0}[x]$ admits exactly one representation of the form $K + M$.

The $D + M$ Construction

Let $T = K + M$ be a semidomain such that K is a yoked semifield and M is a (subtractive) maximal ideal of T. For a subsemidomain D of K, we set $R := D + M$.

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Theorem (Gotti and P., 202?)

Let $T = K + M$ be a semidomain such that K is a yoked semifield and M is a subtractive maximal ideal of T. For a subsemidomain D of K, set $R := D + M$. Every subtractive prime ideal of R is either the contraction of a subtractive prime ideal of T or of the form $P_0 + M$, where P_0 is a subtractive prime ideal of D.

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Thank you!