Prime factorization of ideals in commutative rings, with a focus on Krull rings

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Outline

All rings considered in this talk are commutative with identity.

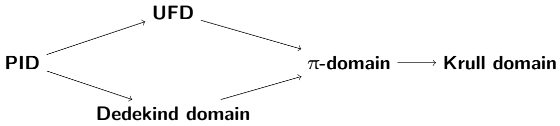
Let R be a ring with total quotient ring $T(R)$.

- An element of R is regular if it is not a zero divisor.

- $Z(R)$ denotes the set of zero divisors of R.
- $reg(R)$ is the set of regular elements of R, so $T(R) = R_{reg(R)}$.

- An ideal of R is regular if it contains a regular element of R.
- An ideal I of R is a Z-ideal if $I \subseteq Z(R)$.

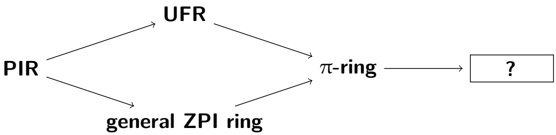
Integral domain Case



- An integral domain D is a π-domain if each nonzero proper principal ideal of D is a finite product of prime ideals.

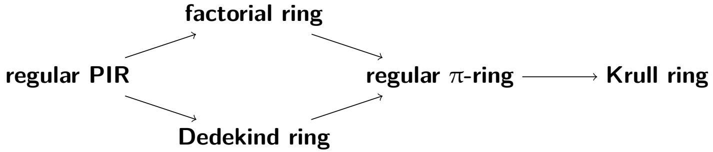
- D is a Krull domain if each nonzero proper principal ideal of D is a finite v-product (t-product) of prime ideals.

Ring with zero divisor case

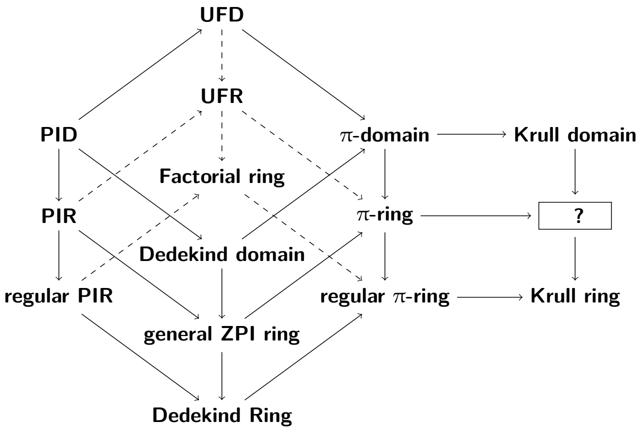


- R is a principal ideal ring (PIR) if each ideal of R is principal.
- R is a (Fletcher's) unique factorization ring (UFR) if each element of R can be written as a finite product of prime elements.
- R is a general ZPI ring if each ideal of R can be written as a finite product of prime ideals.
- R is a π-ring if each principal ideal of R can be written as a finite product of prime ideals.
- (Question) What is a natural generalization of Krull domains to rings with zero divisors ?

Ring characterized by regular elements or ideals case



- R is a regular PIR if each regular ideal of R is principal.
- R is a factorial ring if each regular element of R can be written as a finite product of prime elements.
- R is a Dedekind ring if each regular ideal of R can be written as a finite product of prime ideals.
- R is a regular π-ring if each regular principal ideal of R can be written as a finite product of prime ideals.
- R is a Krull ring if each regular principal ideal of R can be written as a finite v-product of prime ideals.


UFD
UFR
PID
π-domain
Krull domain
Factorial ring
PIR
π-ring
?
Dedekind domain
regular PIR
regular π-ring
Krull ring
general ZPI ring
Dedekind Ring

Krull domains

Let D be an integral domain with quotient field K and $X^1(D)$ be the set of nonzero minimal (i.e., height-one) prime ideals of D. Then D is a Krull domain if

1. $D = \bigcap_{P \in X^1(D)} D_P$,
2. D_P is a DVR for all $P \in X^1(D)$, and
3. each nonzero nonunit of D is contained in only a finitely many prime ideals in $X^1(D)$.

In 1955, Nagata proved that D is a Krull domain if and only if there exists a family Δ of DVRs with quotient field K such that (i) D is the intersection of all rings in Δ and (ii) every nonzero element of D is a unit in all but a finite number of rings in Δ.

The theory of Krull domains was originated by Krull [W. Krull, *Über die Zerlegung der Hauptideale in allgemeinen Ringen*, Math. Ann. **105** (1931), 1-14.].

SPR, PIR and general ZPI ring

• A ring R is said to be a special primary ring (SPR) or a special principal ideal ring (SPIR) if R is a local ring with maximal ideal M such that M is principal and $M^n = (0)$ for some integer $n \geq 1$.

Theorem (*1960, Zariski and Samuel*)

R is a PIR if and only if R is a finite direct sum of PIDs and SPRs.

• In 1940, S. Mori first studied the general ZPI-ring, where the letters ZPI stands for Zerlegung Primideale.

Theorem (*1951, Asano*)

R is a general ZPI-ring if and only if R is a finite direct sum of Dedekind domains and SPRs.

UFR and π-ring

• In 1967, Fletcher introduced the notion of a unique factorization ring (UFR) which is just a UFD in case of integral domains and he showed

Theorem *(1970-1971, C.R. Fletcher)*

R is a UFR if and only if R is a finite direct sum of UFDs and SPRs.

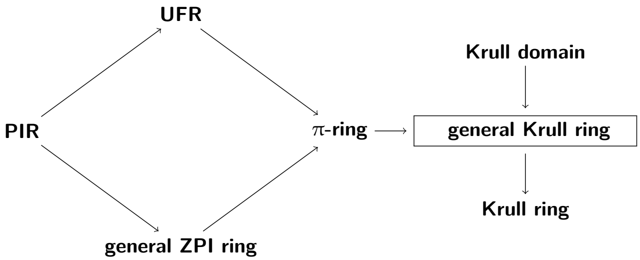
• In 1939, S. Mori gave a complete description of a π-domain.

Theorem *(1940, S. Mori)*

R is a π-ring if and only if R is a finite direct sum of π-domains and SPRs.

Introduction of a general Krull ring

• Inspired by these four types of rings and by the name of general ZPI-rings, we will say that R is *a general Krull ring* if R is a finite direct sum of Krull domains and SPRs, so we have the following implications.



Counterexample

• R is a Krull ring if and only if every regular principal ideal of R can be written as a finite v-product (or t-product) of prime ideals.

However, the next example shows that this is not true of general Krull rings.

Example

Let $R = \mathbb{Z} \times \mathbb{Q}$ be the direct sum of $\mathbb{Z}$ and $\mathbb{Q}$.

1. $\mathbb{Z}$ and $\mathbb{Q}$ are Krull domains, so R is a general Krull ring.
2. If $I = (1, 0)R$, then $I_t = I_v = R$. Hence, I cannot be written as a finite t- nor v-product of prime ideals.

Question and purpose

• It is easy to see that D is a Krull domain if and only if there is a star operation $*$ on D such that each nonzero proper principal ideal of D can be written as a finite $*$-product of prime ideals.

Question

Is there a star operation $*$ on a ring so that a general Krull ring can be characterized as a ring in which each principal ideal can be written as a finite $*$-product of prime ideals?

The purpose of this talk is to answer to Question affirmatively.

Outline

Fractional ideals

An R-submodule of $T(R)$ is called a *Kaplansky fractional ideal*. A Kaplansky fractional ideal of R is regular if it contains a regular element of R.

- $\mathrm{K}(R)$ is the set of Kaplansky fractional ideals of R.
- $F(R)$ is the set of fractional ideals of R (i.e., $I \in F(R)$ if and only if $I \in \mathrm{K}(R)$ and $dI \subseteq R$ for some $d \in reg(R)$), so $F(R) \subseteq \mathrm{K}(R)$.
- An (integral) ideal of R is a fractional ideal of R that is contained in R.

Definition of star operations

- A mapping $*\colon \mathrm{K}(R) \to \mathrm{K}(R)$, given by $I \mapsto I_*$, is a *star operation* on R if the following four conditions are satisfied for all $I, J \in \mathrm{K}(R)$ and $a \in T(R)$:

  1. $R_* = R$,
  2. $aI_* \subseteq (aI)_*$, and equality holds when a is regular.
  3. $I \subseteq I_*$, and $I \subseteq J$ implies that $I_* \subseteq J_*$.
  4. $(I_*)_* = I_*$.

- For all $I \in \mathrm{K}(R)$, let

$$I_{*_f} = \bigcup \{J_* \mid J \in \mathrm{K}(R) \text{ is finitely generated and } J \subseteq I\}.$$

Then $*_f$ is also a star operation on R.

- The star operation $*$ is said to be *of finite type* if $* = *_f$, and $*$ is said to be *reduced* if $(0)_* = (0)$. Clearly, $*_f$ is of finite type, and $*$ is reduced if and only if $*_f$ is reduced.

$*$-ideals

- An $I \in \mathrm{K}(R)$ is a *$*$-ideal* if $I_* = I$. A $*$-ideal I is *of finite type* if $I = J_*$ for some finitely generated subideal J of I. A $*$-ideal is a *maximal $*$-ideal* if it is maximal among proper integral $*$-ideals.
- If $*$ is a star operation of finite type, then
  1. a prime ideal minimal over an integral $*$-ideal is a $*$-ideal,
  2. a proper integral $*$-ideal is contained in a maximal $*$-ideal, and
  3. a maximal $*$-ideal is a prime ideal.
- Let $*_1$ and $*_2$ be star operations on R. We say that $*_1 \leq *_2$ if $I_{*_1} \subseteq I_{*_2}$ for all $I \in \mathrm{K}(R)$, equivalently, $(I_{*_1})_{*_2} = (I_{*_2})_{*_1} = I_{*_2}$.

The d-, v- and t-operation

- The identity function $d : K(R) \to K(R)$ is a star operation.

- For $I \in \mathrm{K}(R)$, let

$$I^{-1} = (R :_{T(R)} I) = \{x \in T(R) \mid xI \subseteq R\},$$

then $I^{-1} \in \mathrm{K}(R)$. The v- and t-operation are defined by

$$I_v = (I^{-1})^{-1} \text{ for all } I \in \mathrm{K}(R)\,, \qquad \text{and} \qquad t = v_f\,.$$

- It is known that $d \leq *_f \leq *$, $*_f \leq t \leq v$, and $* \leq v$ for any star operation $*$ on R.

$*$-invertibility

- An $I \in \mathrm{K}(R)$ is said to be *invertible* if $II^{-1} = R$.

- As the $*$-operation analog, $I \in K(R)$ is said to be *$*$-invertible* if $(II^{-1})_* = R$.

Proposition

If $*$ is a star operation of finite type, then

1. *every $*$-invertible Kaplansky fractional $*$-ideal is of finite type and a t-invertible t-ideal,*
2. *every $*$-invertible prime $*$-ideal is a maximal t-ideal.*

- It is well known that an invertible ideal is regular, while a $*$-invertible ideal need not be regular.

Outline

Valuation rings

A valuation on a ring R is a mapping v from R onto a totally ordered abelian group G with ∞ adjoined such that

(i) $v(ab) = v(a) + v(b)$,
(ii) $v(x + y) \geq \min\{v(a), v(b)\}$ for all $a, b \in R$, and
(iii) $v(1) = 0$ and $v(0) = \infty$.

• If $R_v = \{x \in R \mid v(x) \geq 0\}$ and $P_v = \{x \in R \mid v(x) > 0\}$, then R_v is a subring of R, P_v is a prime ideal of R_v, and (R_v, P_v) is called a valuation pair of R.

• The valuation v on R was first studied by Manis when R is a ring with zero divisors [*Valuations on a commutative ring*, Proc. Amer. Math. Soc. 20 (1969), 193-198].

• If $G = \mathbb{Z}$ is the additive group of integers, then the valuation on R is called a *rank-one discrete valuation* on R.

Rank-one discrete valuation rings

Let v be a rank-one discrete valuation on $T(R)$ such that

$$R = \{x \in T(R) \mid v(x) \geq 0\} \quad \text{and} \quad P = \{x \in T(R) \mid v(x) > 0\}.$$

1. R is called a *rank-one discrete valuation ring* (rank-one DVR).
2. If P is regular (i.e., P contains a regular element), then reg-ht$P = 1$ (i.e., P is a minimal regular prime ideal).
3. If $T(R)$ is a field, then P is the maximal ideal of R, but this is not true in general.

Definition of Krull rings

We say that R is a *Krull ring* if there exists a family $\{(V_\alpha, P_\alpha) \mid \alpha \in \Lambda\}$ of rank-one discrete valuation pairs of $T(R)$ with associated valuations $\{v_\alpha \mid \alpha \in \Lambda\}$ such that

(i) $R = \bigcap\{V_\alpha \mid \alpha \in \Lambda\}$,

(ii) for each regular $a \in T(R)$, $v_\alpha(a) = 0$ for almost all $\alpha \in \Lambda$ and P_α is a regular ideal for all $\alpha \in \Lambda$.

• Krull ring was introduced by J. Marot (1968), J. Huckaba [Integral closure of a Noetherian ring, Trans. Amer. Math. Soc. 220 (1976), 159-666], and Kennedy [*Krull Rings*, Pacific J. Math. **89** (1980), 131-136].

Marot rings

R is a Marot ring if each regular ideal of R is generated by a set of regular elements in R, which was introduced by J. Marot (1969).

Example

A ring R is a Marot ring if R is one of the followings.

1. R is an integral domain.
2. R is a Noetherian ring.
3. $\dim T(R) = 0$.
4. R is an overring of a Marot ring.
5. R is a general Krull ring.

D. Portelli and W. Spangher also studied Krull rings with additional assumption that the rings are Marot [*Krull rings with zero divisors*, Comm. Algebra **11** (1983), 1817-1851].

Characterizations of Krull rings

- $X_r^1(R)$ is the set of minimal regular prime ideals of R.
- $R_{[P]} = \{z \in T(R) \mid zx \in R$ for some $x \in R \setminus P\}$.

Theorem

R is a Krull ring if and only if R satisfies the followings;

1. $R = \bigcap_{P \in X_r^1(R)} R_{[P]}$,
2. $(R_{[P]}, [P]R_{[P]})$ *is a rank-one DVR for all* $P \in X_r^1(R)$, *and*
3. *each regular element of R is contained in only finitely many prime ideals in* $X_r^1(R)$.

This was proved by D. Portelli and W. Spangher in Marot Krull ring case (1983) and by Alajbegović and Osmanagić, in general case [*Essential valuations of Krull rings with zero divisors*, Comm. Algebra **18** (1990), 2007-2020].

Prime factorization of ideals I

1. In 1935, Krull stated (without proof) that D is a Krull domain if and only if each v-ideal I of D is a unique finite v-product of height-one prime ideals of D, i.e., $I = (P_1^{e_1} \cdots P_n^{e_n})_v$ for some distinct height-one prime ideals $P_1, \ldots, P_n$ and positive integers $e_1, \ldots, e_n$ such that the expression $I = (P_1^{e_1} \cdots P_n^{e_n})_v$ is unique [*Idealtheorie*, Ergebnisse der Math. und ihrer Grenz. vol.4, No.3, Berlin, Julius Splinger, 1935].

2. In 1963, Nishimura showed that D is a Krull domain if and only if each v-ideal of D is a unique finite v-product of height-one prime ideals of D, if and only if D is a completely integrally closed Mori domain [*Unique factorization of ideals in the sense of quasi-equality*, J. Math. Kyoto Univ. **3** (1963), 115-125].

Prime factorization of ideals II

1. In 1968, Tramel showed that D is a Krull domain if and only if each proper principal ideal of D can be written as a finite v-product of prime ideals [*Factorization of principal ideals in the sense of quasi-equality*, Doctoral Dissertation, Louisiana State University, 1968], which also shows that the uniqueness of Nishimura's result is superfluous.
2. In 1972, Levitz showed that D is a Krull domain if and only if each nonzero proper principal ideal of D can be written as a finite t-product of prime ideals, if and only if each nonzero t-ideal of D is a finite t-product of height one prime ideals of D [*A characterization of general Z.P.I.-rings*, Proc. Amer. Math. Soc. **32** (1972), 376-380].

Prime factorization of ideals IV

Theorem

The following statements are equivalent for a ring R.

1. *R is a Krull ring.*
2. *Every regular v-ideal I of R is a v-product of prime ideals, i.e., $I = (P_1 \cdots P_n)_t$ for some prime ideals $P_1, \dots, P_n$.*
3. *Every regular t-ideal is a t-product of prime ideals.*
4. *Every regular principal ideal is a v-product of prime ideals.*
5. *Every regular principal ideal is a t-product of prime ideals.*

This was proved by Kang for Marot ring case in [*A characterization of Krull rings with zero divisors*, J. Pure Appl. Algebra **72** (1991), 33-38] and for general case in [*Characterizations of Krull rings with zero divisors*, J. Pure Appl. Algebra **146** (2000), 283-290].

Outline

The w-operation I

• Let R be an integral domain. A nonzero finitely generated ideal I of R is called a *GV-ideal* if $I^{-1} = R$, where GV stands for Glaz and Vasconselos, and we denote by $GV(R)$ the set of GV-ideals of R.

• The w-operation on R is a star operation defined by

$$I_w = \{x \in T(R) \mid xJ \subseteq I \text{ for some } J \in GV(R)\}$$

for all $I \in \mathrm{K}(R)$. Then w is of finite type, t-Max$(R) = w$-Max(R), $w \leq t$, $(0)_w = (0)$, and $I_w = \bigcap_{P \in t\text{-Max}(R)} IR_P$ for all $I \in F(R)$.

• The w-operation was introduced by Hedstrom and Houston [Some remarks on star operations, J. Pure Appl. Algebra 18 (1980), 37-44] under the name of an F_∞-operation.

• The notation of w-operation was first used by R. McCasland and F.Wang [On w-modules over strong Mori domains, Comm. Algebra 25 (1997), 1285-1306].

The w-operation II

- The w-operation was generalized to rings with zero divisors by Yin, Wang, Zhu, and Chen [w-modules over commutative rings, J. Korean Math. Soc. 48 (2011), 207-222].

- A finitely generated ideal J of R is called a *GV-ideal* if the homomorphism $\varphi\colon R \to Hom_R(J, R)$ given by $\varphi(r)(a) = ra$ is an isomorphism.

- If J is regular, then $Hom_R(J, R) = J^{-1}$, so φ is an isomorphism if and only if $J^{-1} = R$.

- The w-operation on R defined by, for all $A \in \mathrm{K}(R)$,

$$A_w = \{x \in T(R) \mid xJ \subseteq A \text{ for some } J \in GV(R)\}$$

is a reduced star operation of finite type.

- F.G. Wang and H. Kim, *Foundations of Commutative Rings and Their Modules*, Algebra and Applications, vol.22, Singapore, Springer, 2016.

The u-operation I

- Let $\mathrm{rGV}(R) = \{J \in GV(R) \mid J \text{ is regular}\}$. Then $rGV(R)$ is a multiplicative set of regular ideals of R.

- For each $I \in \mathrm{K}(R)$, let

$$I_u = \{x \in T(R) \mid xJ \subseteq I \text{ for some } J \in rGV(R)\}.$$

Then $I_u \in K(R)$ and the map $u\colon \mathrm{K}(R) \to \mathrm{K}(R)$, given by $I \mapsto I_u$, is a reduced star operation of finite type on R.

The u-operation II

Theorem

The following conditions hold for all $a \in T(R)$ and $I, J \in \mathrm{K}(R)$:

1. $R_u = R$.
2. $aI_u \subseteq (aI)_u$, *and equality holds when a is regular.*
3. $I \subseteq I_u$, *and $I \subseteq J$ implies $I_u \subseteq J_u$.*
4. $(I_u)_u = I_u$.
5. $(0)_u = (0)$.
6. $I_u = \bigcup\{(I_0)_u \mid I_0 \in \mathrm{K}(R),\ I_0 \subseteq I,$ *and I_0 is finitely generated*$\}$.

Thus, the map $u\colon \mathrm{K}(R) \to \mathrm{K}(R)$, given by $I \mapsto I_u$, is a reduced star operation of finite type on R.

The u-operation III

• A ring R satisfies *Property(A)* if each finitely generated Z-ideal $I \subseteq Z(R)$ has a nonzero annihilator. Then R has Property(A) if and only if $T(R)$ has Property(A).

• The class of rings with Property(A) includes Noetherian rings, the polynomial ring, integral domains, and general Krull rings.

Proposition

1. $u \leq w$.
2. $I_u = I_w$ *for all regular* $I \in \mathrm{K}(R)$.
3. *If* R *satisfies Property(A), then* $u = w$ *on* R.

Characterizations of general Krull rings I

Theorem

The following statements are equivalent for a ring R.

1. *R is a general Krull ring.*
2. *Each principal ideal of R is a finite u-product of prime ideals.*
3. *Each integral u-ideal of R is a finite u-product of prime ideals.*
4. *R is a Krull ring, $\dim(T(R)) = 0$, and each minimal prime ideal of R is a principal ideal.*
5. *R is a Krull ring, $\dim(T(R)) = 0$, and the zero element of R is a finite product of prime elements.*

Characterizations of general Krull rings II

• Let R be a general Krull ring. Then $u = w$ on R because R satisfies Property(A). Hence, each principal ideal of R is a finite w-product of prime ideals.

Corollay

The following statements are equivalent for a ring R.

1. *R is a general Krull ring.*
2. *Each principal ideal of R is a finite w-product of prime ideals.*
3. *Each integral w-ideal of R is a finite w-product of prime ideals.*

Corollay

If R is a general Krull ring, then $T(R)$ is a zero-dimensional PIR.

When is a Krull ring a general Krull ring ?

Question: If R is a Krull ring such that $T(R)$ is a zero-dimensional PIR, then R is a general Krull ring ?

Example

Let V be a rank-two discrete valuation ring, Q be a primary ideal of V such that $ht(\sqrt{Q}) = 1$ and $Q \subsetneq \sqrt{Q}$, and $R = V/Q$ be the factor ring of V modulo Q. Then the following conditions hold.

1. *$T(R)$ is an SPR.*
2. *R is a Krull ring but not a general Krull ring.*

Theorem

Let R be a Krull ring such that $T(R)$ is a zero-dimensional PIR. Then R is a general Krull ring if and only if R_P is a DVR for all $P \in X_r^1(R)$.

Mori-Nagata Theorem I

Theorem

The integral closure of a Noetherian domain is a Krull domain.

Proof.

1. This was conjectured by Krull [*Idealtheorie*, Ergebnisse der Math. und ihrer Grenz. vol.4, No.3, Berlin, Julius Splinger, 1935].
2. The local case was proved by Mori [*On the integral closure of an integral domain*, Mem. Coll. Sci. Univ. Kyoto Ser. A. Math. **27** (1953), 249-256].
3. The general case was proved by Nagata [*On the derived normal rings of Noetherian integral domains*, Mem. Coll. Sci. Univ. Kyoto Ser. A. Math. **29** (1955), 293-303].

□

Mori-Nagata Theorem II

- Krull-Akizuki theorem says that every overring of a one-dimensional Noetherian domain is Noetherian.
- In 1953, Nagata constructed
  1. a two-dimensional Noetherian domain R such that there is a non-Noetherian ring between R and its integral closure and
  2. a three-dimensional Noetherian domain whose integral closure is not Noetherian.

Theorem

The integral closure of a two-dimensional Noetherian domain is Noetherian.

This was proved by Mori for local rings [*On the integral closure of an integral domain*, Mem. Coll. Sci. Univ. Kyoto Ser. A. Math. **27** (1953), 249-256], and generalized by Nagata [*On the derived normal rings of Noetherian integral domains*, Mem. Coll. Sci. Univ. Kyoto Ser. A. Math. **29** (1955), 293-303].

Mori-Nagata theorem III

- R is r-Noetherian if each regular ideal of R is finitely generated.

- In [*The integral closure of a Noetherian ring*, Trans. Amer. Math. Soc. **220** (1976), 159-166], Huckaba constructed an n-dimensional Noetherian ring whose integral closure is not Noetherian for any integer $n \geq 1$, and he showed

Theorem

1. *The integral closure of a Noetherian ring is a Krull ring.*
2. *If R is a Noetherian ring with $\dim(R) \leq 2$, then the integral closure of R is r-Noetherian.*

Mori-Nagata theorem IV

• Mori-Nagata theorem has been generalized to Non-Noetherian rings with zero divisors.

Theorem

Let $\bar{R}$ be the integral closure of an r-Noetherian ring R.

1. *$\bar{R}$ is a Krull ring.*
2. *If r-dim$(R) \leq 2$, then $\bar{R}$ is an r-Noetherian ring.*

This theorem has been proved by a series of papers by Kang and Chang (1993, 1999, 2002, and 2023).

Mori-Nagata theorem V

The next example shows that the integral closure of a Noetherian ring R need not be general Krull even though $T(R)$ is an SPR.

Example

Let $\mathbb{Q}$ be the field of rational numbers, $\mathbb{Q}[X]$ be the polynomial ring over $\mathbb{Q}$, and $A = \mathbb{Q}[X]/(X^2)$; so A is an SPR. Let $\mathrm{m} = (X)/(X^2)$, Y be an indeterminate over A, and $R = A[Y]$. Then R is a one-dimensional Noetherian ring and $T(R) = A(Y)$, so $T(R)$ is an SPR. But, note that $N(A) = \mathrm{m}$ and $N(R) = N(A)[Y]$, so $N(R) = \mathrm{m}[Y]$ is a prime ideal of R. Hence, if $\overline{R}$ is the integral closure of R, then $N(\overline{R}) = \mathrm{m}T(R)$, which is a nonzero prime ideal of $\overline{R}$, but since $N(\overline{R}) \cap R = \mathrm{m}[Y]$, $N(\overline{R})$ is not a maximal ideal of $\overline{R}$. Thus, $\overline{R}$ is a Krull ring but $\overline{R}$ is not a general Krull ring.

Mori-Nagata theorem VI

- R is a Noetherian ring and $\overline{R}$ is the integral closure of R.

Theorem

If R is integrally closed, then R is a general Krull ring if and only if $T(R)$ is a PIR.

Corollay

If $\dim R \leq 2$, then $\overline{R}$ is a general Krull ring if and only if $\overline{R}$ is a Noetherian ring and $T(R)$ is a PIR.

Nagata ring I

• Let R be a ring, X be an indeterminate over R, $R[X]$ be the polynomial ring over R, and

$$N_* = \{f \in R[X] \mid c(f)_* = R\}$$

for $* = d, u, w, v$, so $N_d \subseteq N_u \subseteq N_w \subseteq N_v$.

Proposition

1. *N_u is a saturated multiplicative set of $R[X]$.*
2. *Each element of N_u is regular. Hence, $R[X]_{N_u}$ is an overring of $R[X]$.*
3. $Max(R[X]_{N_u}) = \{P[X]_{N_u} \mid P \in u\text{-}Max(R)\}$.

• It is clear that if R is an integral domain, then $N_u = N_w = N_v$ and $R[X]_{N_u}$ is the (t-)Nagata ring of R.

Nagata ring II

Theorem

The following are equivalent for a ring R with Property(A).

1. *R is a Krull ring.*
2. *$R[X]_{N_u}$ is a Krull ring.*
3. *$R[X]_{N_u}$ is a regular π-ring.*
4. *$R[X]_{N_u}$ is a factorial ring.*
5. *$R[X]_{N_u}$ is a Dedekind ring.*
6. *$R[X]_{N_u}$ is a regular PIR.*

Corollay

If R is the integral closure of a Noetherian ring, then $R[X]_{N_u}$ is a regular PIR.

Nagata ring III

- Let $R(X) = R[X]_{N_d}$. Then $R(X)$ is called the Nagata ring of R and $R(X) \subseteq R[X]_{N_u}$.

Theorem

The following statements are equivalent for a ring R.

1. *R is a general Krull ring.*
2. *$R[X]_{N_u}$ is a general Krull ring.*
3. *$R[X]_{N_u}$ is a π-ring.*
4. *$R[X]_{N_u}$ is a UFR.*
5. *$R[X]_{N_u}$ is a general ZPI-ring.*
6. *$R[X]_{N_u}$ is a PIR.*
7. *$R(X)$ is a general Krull ring.*

u-Almost Dedekind rings and general Krull rings I

• A ring is a *u-Noetherian ring* if it satisfies the ascending chain condition on its integral u-ideals.

• An integral domain D is a Krull domain if and only if D is a u-Noetherian domain (strong Mori domain) and R_P is a DVR for all maximal u-ideals P of R.

Theorem

The following statements are equivalent for a ring R.

1. *R is a general Krull ring.*
2. *R is a u-Noetherian ring such that R_P is a DVR or an SPR for all maximal u-ideals P of R.*

u-Almost Dedekind rings and general Krull rings II

Corollay

A ring R is a general ZPI-ring if and only if R is Noetherian and R_M is a DVR or an SPR for all $M \in Max(R)$.

• We will say that R is an almost Dedekind ring (resp., a u-almost Dedekind ring) if R_M is a DVR or an SPR for all maximal ideals (resp., maximal u-ideals) M of R.

• Hence, R is a general ZPI-ring (resp., general Krull ring) if and only if R is a Noetherian almost Dedekind ring (resp., a u-almost Dedekind u-Noetherian ring).

u-Almost Dedekind rings and general Krull rings III

Corollary

The following statements are equivalent for a ring R.

1. *R is a general Krull ring.*
2. *R satisfies the following conditions.*
   1. $R = \bigcap_{P \in X_r^1(R)} R_{[P]}$.
   2. *R_P is a DVR for all $P \in X_r^1(R)$ and R_P is an SPR for all prime Z-ideals P of R.*
   3. *Each principal ideal of R has a finite number of minimal prime ideals.*
3. *R is a u-almost Dedekind ring in which each principal ideal has a finite number of minimal prime ideals.*

Outline

Juett's general w-ZPI ring I

• In [General w-ZPI-rings and a tool for characterizing certain classes of monoid rings, Comm. Algebra 51 (2023), 1117-1134], Juett introduced the notion of general w-ZPI rings.

• Juett called R a general w-ZPI ring if every proper w-ideal of R is a finite w-product of prime w-ideals. Then, among other things, he proved

Theorem (J.R. Juett, 2023, Comm. Algebra)

The following statements are equivalent.

1. *R is a general w-ZPI ring.*
2. *R is a finite direct product of Krull domains and SPRs.*
3. *$R[X]_{N_w}$ is an Euclidean ring.*

Therefore, Juett's general w-ZPI ring is exactly the general Krull ring of this talk.

Juett's general w-ZPI ring II

• Let S be a commutative cancellative additive monoid and $R[S]$ be the semigroup ring of S over R. Juett has studied several factorization properties of $R[S]$ including Dedekind rings, π-rings, UFRs, and general w-ZPI rings [J1, J2, J3].

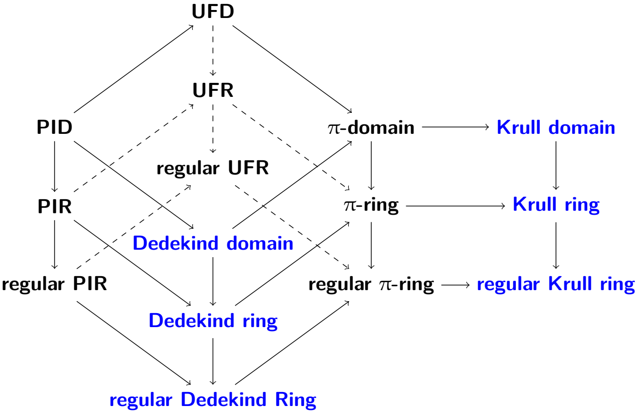
[J1] J.R. Juett, *General w-ZPI-rings and a tool for characterizing certain classes of monoid rings*, Comm. Algebra 51 (2023), 1117-1134.

[J2] J.R. Juett, C.P. Mooney, and L.W. Ndungu, *Unique factorization of ideals in commutative rings with zero divisors*, Comm. Algebra 49 (2021), 2101-2125.

[J3] J.R. Juett, C.P. Mooney, and R.D. Roberts, *Unique factorization properties in commutative monoid rings with zero divisors*, Semigroup Forum 102 (2021), 674-696.

Outline

Renaming ?


UFD
UFR
PID
π-domain
Krull domain
regular UFR
PIR
π-ring
Krull ring
Dedekind domain
regular PIR
regular π-ring
regular Krull ring
Dedekind ring
regular Dedekind Ring

What is a star operation ?

• The idea of localization comes from algebraic geometry. The localization at a point p allows us to focus on only rational functions that are well-defined at the point p.

• A star operation is a similar tool for studying the ideal factorization properties of commutative rings in the sense that we are just interested in ideals that we certainly have in mind.

• For example, in Krull domains, every nonzero proper principal ideal is a unique finite v-product of height-one prime ideals. Hence, when we study the ideal factorization of Krull domains, it is enough to look into the height-one prime ideals.

• Localization.

All of the results in this talk appear in

G.W. Chang and J.S. Oh, Prime factorization of ideals in commutative rings, with a focus on Krull rings. J. Korean Math. Soc. 60 (2023), no. 2, 407–464.

Thank you !!