Representation and Subrepresentation of Γ-Monoids

Flor de May C. Lañohan
and
Jocelyn P. Vilela, Ph.D.

Department of Mathematics and Statistics
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OUTLINE OF THE PRESENTATION

- Background of the Study
- Objectives
- Basic Concepts
- Results
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to understand algebraic structures by transforming their elements into matrices.

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"GROUPS"

to understand algebraic structures by transforming their elements into matrices.

Background of the study

Representation of Monoids

2015 - Steinberg studied the representation theory of finite monoids

➡ Character Theory of monoids over an arbitrary field

Background of the study

Γ-Monoid

2020 - Hazrat and Li defined the "Γ-monoid" as a monoid with the group Γ acting on it.

➡ Talented monoid T_E ($\mathbb{Z}$-monoid)

2022 - action via **monoid automorphism**

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2022 - action via **monoid automorphism**

The focus of this presentation is the representation of Γ-monoids and its subrepresentation.

This study has the following objectives:

- to introduce the concept of the Γ-invariant and subrepresentation of a representation;
- to show that a subrepresentation is also a representation;
- to demonstrate that a restriction map to the kernel, image, and inverse image of a Γ-linear map are subrepresentations.

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Γ-MONOIDS

Definition 1 [2]

Let M be a monoid and Γ a group.
M is said to be a Γ-*monoid* if there is an action of Γ on M via monoid automorphism.

For $\alpha \in \Gamma$ and $a \in M$, the action of α on a shall be denoted by ${}^{\alpha}a$.

$$\boxed{{}^{\alpha}(a+b) = {}^{\alpha}a +^{\alpha} b}$$

Example 2

Let $\Gamma = \mathbb{Z}$ be the group of integers under addition and $\mathbb{C}$ be the set of complex numbers.

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Note that $\mathbb{C}$ is a group under addition. Thus, it is a monoid.

Consider the mapping $\Gamma \times \mathbb{C} \to \mathbb{C}$ given by

$$(x, a + bi) \mapsto e^x a + e^x bi$$

for all $x \in \Gamma$ and $a + bi \in \mathbb{C}$.

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(iii) $$\begin{aligned} {}^{x}[(a+bi)+(c+di)] &= e^{x}(a+c) + e^{x}(b+d)i \\ &= {}^{x}(a+bi) + {}^{x}(c+di), \end{aligned}$$

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the mapping is an action of a group Γ on $\mathbb{C}$.

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Hence, $\mathbb{C}$ is a Γ-monoid.

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Let $\Gamma = \mathbb{Z}$ and $M_2(\mathbb{C})$ be a square matrix with entries from $\mathbb{C}$.

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Note that $M_2(\mathbb{C})$ is a monoid under matrix addition.

Define the mapping $\Gamma \times M_2(\mathbb{C}) \to M_2(\mathbb{C})$ by

$$\left(x, \begin{bmatrix} a_1 + b_1 i & a_2 + b_2 i \\ a_3 + b_3 i & a_4 + b_4 i \end{bmatrix}\right) \mapsto \begin{bmatrix} e^x(a_1 + b_1 i) & a_2 + b_2 i \\ a_3 + b_3 i & e^x(a_4 + b_4 i) \end{bmatrix}$$

for all $x \in \Gamma$ and $\begin{bmatrix} a_1 + b_1 i & a_2 + b_2 i \\ a_3 + b_3 i & a_4 + b_4 i \end{bmatrix} \in M_2(\mathbb{C})$.

$$\text{Since for all } x, y \in \Gamma \text{ and } \begin{bmatrix} a_1 + b_1 i & a_2 + b_2 i \\ a_3 + b_3 i & a_4 + b_4 i \end{bmatrix} \in M_2(\mathbb{C}),$$

$$\text{(i)} \quad {}^{0}\begin{bmatrix} a_1 + b_1 i & a_2 + b_2 i \\ a_3 + b_3 i & a_4 + b_4 i \end{bmatrix} = \begin{bmatrix} e^0(a_1 + b_1 i) & a_2 + b_2 i \\ a_3 + b_3 i & e^0(a_4 + b_4 i) \end{bmatrix}$$

$$= \begin{bmatrix} a_1 + b_1 i & a_2 + b_2 i \\ a_3 + b_3 i & a_4 + b_4 i \end{bmatrix}$$

(ii)
$$
\begin{aligned}
{}^{x+y}\begin{bmatrix} a_1+b_1i & a_2+b_2i \\ a_3+b_3i & a_4+b_4i \end{bmatrix}
&= \begin{bmatrix} e^{(x+y)}(a_1+b_1i) & a_2+b_2i \\ a_3+b_3i & e^{(x+y)}(a_4+b_4i) \end{bmatrix} \\
&= \begin{bmatrix} e^{x}e^{y}(a_1+b_1i) & a_2+b_2i \\ a_3+b_3i & e^{x}e^{y}(a_4+b_4i) \end{bmatrix} \\
&= {}^{x}\left({}^{y}\begin{bmatrix} a_1+b_1i & a_2+b_2i \\ a_3+b_3i & a_4+b_4i \end{bmatrix}\right),
\end{aligned}
$$

(iii) $$ {}^{x}\left(\begin{bmatrix} a_1+b_1i & a_2+b_2i \\ a_3+b_3i & a_4+b_4i \end{bmatrix} + \begin{bmatrix} c_1+d_1i & c_2+d_2i \\ c_3+d_3i & c_4+d_4i \end{bmatrix}\right) $$
$$ = \begin{bmatrix} e^x((a_1+b_1i)+(c_1+d_1i)) & (a_2+b_2i)+(c_2+d_2i) \\ e^x(a_3+b_3i) & (a_4+b_4i)+(c_4+d_4i) \end{bmatrix} $$
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$$= {}^{x}\begin{bmatrix} a_1+b_1i & a_2+b_2i \\ a_3+b_3i & a_4+b_4i \end{bmatrix} + {}^{x}\begin{bmatrix} c_1+d_1i & c_2+d_2i \\ c_3+d_3i & c_4+d_4i \end{bmatrix}$$

the mapping is an action of a group Γ on $M_2(\mathbb{C})$.

(iii) $$^{x}\left(\begin{bmatrix} a_1+b_1i & a_2+b_2i \\ a_3+b_3i & a_4+b_4i \end{bmatrix} + \begin{bmatrix} c_1+d_1i & c_2+d_2i \\ c_3+d_3i & c_4+d_4i \end{bmatrix}\right)$$

$$= \begin{bmatrix} e^{x}((a_1+b_1i)+(c_1+d_1i)) & (a_2+b_2i)+(c_2+d_2i) \\ e^{x}(a_3+b_3i) & (a_4+b_4i)+(c_4+d_4i) \end{bmatrix}$$

$$= {}^{x}\begin{bmatrix} a_1+b_1i & a_2+b_2i \\ a_3+b_3i & a_4+b_4i \end{bmatrix} + {}^{x}\begin{bmatrix} c_1+d_1i & c_2+d_2i \\ c_3+d_3i & c_4+d_4i \end{bmatrix}$$

the mapping is an action of a group Γ on $M_2(\mathbb{C})$.

Hence, $M_2(\mathbb{C})$ is a Γ-monoid.

Definition 4 [2]

Let M_1, M_2 be monoids and Γ be a group acting on M_1 and M_2.

A Γ-*monoid homomorphism* is a monoid homomorphism $\rho : M_1 \to M_2$ that respects the action of Γ, that is,

$$\rho\left({}^{\alpha}a\right) = {}^{\alpha}\rho(a)$$

for all $a \in M_1$.

Example 5

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$$(x, a + bi) \mapsto e^x a + bi$$

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NOTE: Let $\Gamma = \mathbb{Z}$. In Example 2 and 3, $\mathbb{C}$ and $M_2(\mathbb{C})$ are Γ-monoids via the action $\Gamma \times \mathbb{C} \to \mathbb{C}$ given by

$$(x, a+bi) \mapsto e^x a + bi$$

and $\Gamma \times M_2(\mathbb{C})$ by

$$\left(x, \begin{bmatrix} a_1+b_1i & a_2+b_2i \\ a_3+b_3i & a_4+b_4i \end{bmatrix}\right) \mapsto \begin{bmatrix} e^x(a_1+b_1i) & a_2+b_2i \\ a_3+b_3i & e^x(a_4+b_4i) \end{bmatrix}$$

respectively, for all $x \in \Gamma$, $a+bi \in \mathbb{C}^*$, and $\begin{bmatrix} a_1+b_1i & a_2+b_2i \\ a_3+b_3i & a_4+b_4i \end{bmatrix} \in M_2(\mathbb{C})$.

Consider the mapping $\rho : \mathbb{C} \to M_2(\mathbb{C})$ given by

$$a + bi \mapsto \begin{bmatrix} a+bi & 0 \\ 0 & a+bi \end{bmatrix}$$

for all $a + bi \in \mathbb{C}$.

Let $x \in \Gamma$ and $a_1 + b_1 i, a_2 + b_2 i \in \mathbb{C}$. Then

$$\rho(1+0i) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$\begin{aligned}
&\rho(a_1 + b_1 i)\rho(a_2 + b_2 i) \\
&= \begin{bmatrix} a_1 + b_1 i & 0 \\ 0 & a_1 + b_1 i \end{bmatrix} \begin{bmatrix} a_2 + b_2 i & 0 \\ 0 & a_2 + b_2 i \end{bmatrix} \\
&= \begin{bmatrix} (a_1 + b_1 i)(a_2 + b_2 i) & 0 \\ 0 & (a_1 + b_1 i)(a_2 + b_2 i) \end{bmatrix} \\
&= \rho((a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1) i) \\
&= \rho((a_1 + b_1 i)(a_2 + b_2 i)).
\end{aligned}$$

Also,

$$\begin{aligned}
\rho({}^{x}(a_1 + b_1 i)) &= \rho(e^x(a_1 + b_1 i)) \\
&= \rho(e^x a_1 + e^x b_1 i) \\
&= \begin{bmatrix} e^x a_1 + e^x b_1 i & 0 \\ 0 & e^x a_1 + e^x b_1 i \end{bmatrix} \\
&= \begin{bmatrix} e^x (a_1 + b_1 i) & 0 \\ 0 & e^x (a_1 + b_1 i) \end{bmatrix} \\
&= {}^{x}[\rho(a_1 + b_1 i)].
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&= \begin{bmatrix} e^x(a_1+b_1i) & 0 \\ 0 & e^x(a_1+b_1i) \end{bmatrix} \\
&= {}^{x}[\rho(a_1+b_1i)].
\end{aligned}$$

Hence, ρ is a Γ-monoid homomorphism.

Definition 6

Let M and $M_r(K)$ be a Γ-monoid where K is a field. A *representation* of M over K is a Γ-monoid homomorphism $\varphi : M \to M_r(K)$.

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Let M and $M_r(K)$ be a Γ-monoid where K is a field. A *representation* of M over K is a Γ-monoid homomorphism $\varphi : M \to M_r(K)$.

(i) φ is a monoid homomorphism

(ii) $\varphi\left({}^{\alpha}a\right) = {}^{\alpha}\varphi(a)$

The mapping $\rho : \mathbb{C}^* \to M_2(\mathbb{C})$ given by

$$a + bi \mapsto \begin{bmatrix} a+bi & 0 \\ 0 & a+bi \end{bmatrix}$$

for $a + bi \in \mathbb{C}^*$, in Example 5, is a representation.

Definition 7

Let M be a Γ-monoid and $\varphi : M \to M_r(K)$ be a representation of M over a field K.

A subspace V of K^r is *Γ-invariant* if for all $\alpha \in \Gamma, m \in M$, and $v \in V$,

$$\varphi({}^{\alpha}m) \cdot v \in V.$$

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Consider the representation $\rho : \mathbb{C} \to M_2(\mathbb{C})$ given by

$$a + bi \mapsto \begin{bmatrix} a+bi & 0 \\ 0 & a+bi \end{bmatrix}$$

for $a + bi \in \mathbb{C}$.

Take

$$V = \{(-s+ri, r+si) \mid r+si, -s+ri \in \mathbb{C}\} \subset \mathbb{C}^2.$$

Thus, V is a proper subspace of $\mathbb{C}^2$.

Let $v = (-s + ri, r + si)$.

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$$\begin{aligned}
&\rho({}^{x}(a + bi)) \cdot v \\
&= \rho(e^x a + e^x bi) \cdot \begin{bmatrix} -s + ri \\ r + si \end{bmatrix} \\
&= \begin{bmatrix} e^x a + e^x bi & 0 \\ 0 & e^x a + e^x bi \end{bmatrix} \begin{bmatrix} -s + ri \\ r + si \end{bmatrix} \\
&= \begin{bmatrix} -(e^x as + e^x br) + (e^x ar - e^x bs)i & 0 \\ 0 & (e^x ar - e^x bs) + (e^x as + e^x br)i \end{bmatrix}
\end{aligned}$$

$\in V$. Hence, V is Γ-invariant.

representation $\varphi : M \to M_r(K)$

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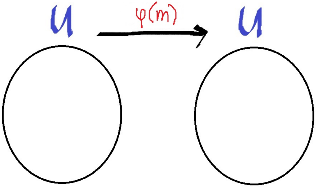

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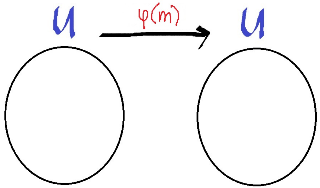

Suppose V is a subspace of U, where $dim(V) = s$.

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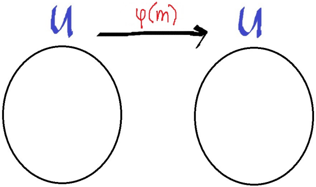

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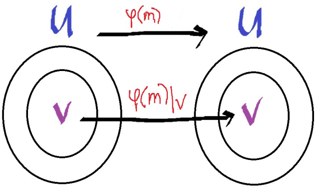

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$M_s(K) \cong Hom(V, V)$

$$\begin{aligned}\varphi_V &: M \to M_s(K)\\ \varphi_V &: M \to Hom(V, V)\\ & m \mapsto \varphi(m)\mid_V : V \to V\end{aligned}$$

Definition 9

Let φ be a representation of a Γ-monoid M over a field K and V be a subspace of K^r.

We say that $\varphi_V : M \to M_s(K)$ is a *subrepresentation* of φ if V is Γ-invariant, that is,

$$\varphi({}^{\alpha}m) \cdot v \in V$$

for all $\alpha \in \Gamma, m \in M$, and $v \in V$.

Example 10

In Example 5, $\rho : \mathbb{C} \to M_2(\mathbb{C})$ given by $a+bi \mapsto \begin{bmatrix} a+bi & 0 \\ 0 & a+bi \end{bmatrix}$ for all $a+bi \in \mathbb{C}$ is a representation.

By previous example, $V = \{(-s+ri, r+si) \mid r+si, -s+ri \in \mathbb{C}\}$ is a Γ-invariant.

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By previous example, $V = \{(-s + ri, r + si) \mid r + si, -s + ri \in \mathbb{C}\}$ is a Γ-invariant.

Thus, ρ_V is a subrepresentation.

Proposition 11

A subrepresentation is a representation of Γ-monoid M over a field K.

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(i) $\varphi_V(1) = \varphi(1)|_V = \varphi(1) = I_r,$

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$$
\begin{aligned}
\text{(i)}\quad & \varphi_V(1) = \varphi(1)|_V = \varphi(1) = I_r, \\
\text{(ii)}\quad & \varphi_V(m_1)\varphi_V(m_2) = \varphi(m_1)|_V\, \varphi(m_2)|_V \\
& \qquad = \varphi(m_1 m_2)|_V \\
& \qquad = \varphi_V(m_1 m_2), \text{ and}
\end{aligned}
$$

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&\qquad = \varphi(m_1 m_2)|_V \\
&\qquad = \varphi_V(m_1 m_2), \text{ and} \\
&\text{(iii)} \quad \varphi_V({}^{\alpha}m_1) = \varphi({}^{\alpha}m_1)|_V \\
&\qquad = {}^{\alpha}(\varphi(m_1)|_V) \\
&\qquad = {}^{\alpha}(\varphi_V(m_1)).
\end{aligned}$$

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\text{(iii)}\quad & \varphi_V({}^{\alpha}m_1) = \varphi({}^{\alpha}m_1)|_V \\
& \qquad = {}^{\alpha}(\varphi(m_1)|_V) \\
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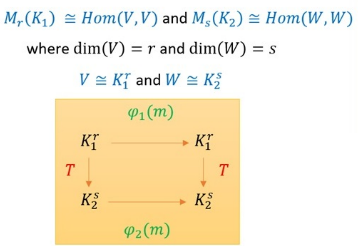
\end{aligned}
$$

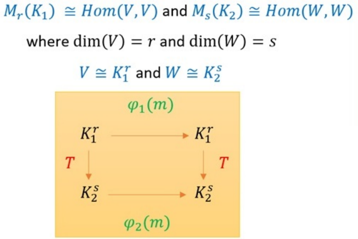
$\therefore \varphi_V$ is a representation

Definition 12



Let K_1 and K_2 be fields, and $\varphi_1 : M \to M_r(K_1)$ and $\varphi_2 : M \to M_s(K_2)$ be representations of a Γ-monoid M over K_1 and K_2, respectively.

Definition 12



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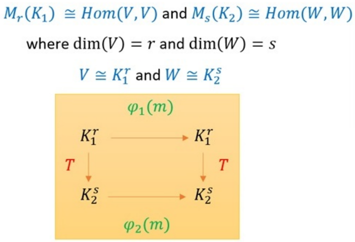

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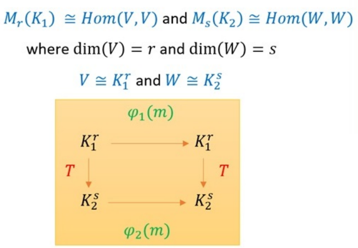

Let K_1 and K_2 be fields, and $\varphi_1 : M \to M_r(K_1)$ and $\varphi_2 : M \to M_s(K_2)$ be representations of a Γ-monoid M over K_1 and K_2, respectively.

A function $T : K_1^r \to K_2^s$ is called a *Γ-linear map* if T is a

(i) linear transformation and

(ii) $T(\varphi_1(^{\alpha}m)k) = \varphi_2(^{\alpha}m)T(k)$

for all $\alpha \in \Gamma, m \in M$, and $k \in K_1$.

Proposition 13

Let K_1 and K_2 be fields, and $\varphi : M \to M_r(K_1)$ and $\psi : M \to M_s(K_2)$ be representations of a Γ-monoid M over K_1 and K_2, respectively. Suppose $T : K_1^r \to K_2^s$ is a Γ-linear map. Then

Proposition 13

Let K_1 and K_2 be fields, and $\varphi : M \to M_r(K_1)$ and $\psi : M \to M_s(K_2)$ be representations of a Γ-monoid M over K_1 and K_2, respectively. Suppose $T : K_1^r \to K_2^s$ is a Γ-linear map. Then

$u \in Ker\, T \Longrightarrow T(u) = 0_{K_2^s}$

$T(\varphi(^{\alpha}m)u) = \psi(^{\alpha}m)T(u) = \psi(^{\alpha}m)0_{K_2^s} = 0_{K_2^s}$

$\Longrightarrow (\varphi(^{\alpha}m)u \in KerT$

$\therefore$ $Ker\, T$ is Γ−invariant

$w \in Im\, T \Longrightarrow w = T(k)$ for some $k \in K_1^r$

$\psi(^{\alpha}m)w = \psi(^{\alpha}m)T(k) = T(\varphi(^{\alpha}m)(k)) \in Im\, T$

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(i) φ_{KerT} is a subrepresentation of φ,

Proposition 13

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$u \in Ker\,T \Longrightarrow T(u) = 0_{K_2^s}$

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$\therefore$ $Ker\,T$ is Γ−invariant

$w \in Im\,T \Longrightarrow w = T(k)$ for some $k \in K_1^r$

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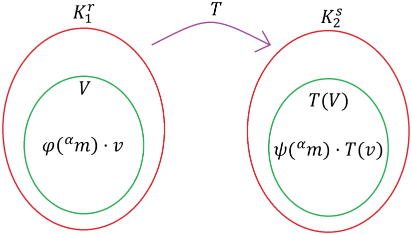
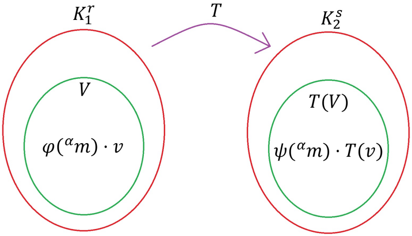
(i) φ_{KerT} is a subrepresentation of φ,

(ii) ψ_{ImT} is a subrepresentation of ψ,



(iii) if V is a Γ-invariant of K_1^r, then $T(V)$ is Γ-invariant subspace of K_2^s, and



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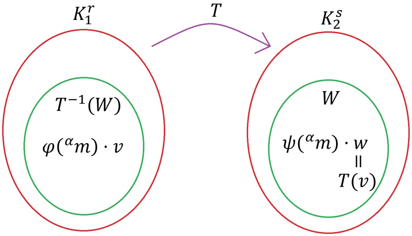

(iv) if W is a Γ-invariant subspace of K_2^s, then $T^{-1}(W)$ is Γ-invariant subspace of K_1^r.

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Thank you all for this wonderful experience! 😁