Pre-Lie algebras

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Conference on Rings
and Factorizations 2023

Graz, 11 July 2023

This talk is dedicated to Matej Brešar

"Life is a journey, it can take you anywhere you choose to go."

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Christina Aguilera

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Christina Aguilera (The Voice Within, 1998)

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Smoktunowicz, Brešar: Pre-Lie algebras

$\Downarrow$

Tomasz Brzeziński: The beauty of ternary operations (heaps and trusses).

In the Garden of Algebra there bloom a number of wonderful algebraic structures.

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(1) M. Cerqua and A. Facchini, Pre-Lie algebras, their multiplicative lattice, and idempotent endomorphisms, to appear in Functor Categories, Module Theory, Algebraic Analysis, and Constructive Methods, A. Martsinkovski Ed., Springer Proc. Math. and Stat., 2023, also available at: arXiv:2301.02627.

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(2) F. Azmy Ebrahim and A. Facchini, Idempotent pre-endomorphisms of algebras, submitted for publication, 2023, available at: arXiv:2304.05079.

Herstein

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Matej

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(1) (alternativity, or anticommutativity:) $[x, x] = 0$ for every $x \in R$; and
(2) (the Jacobi identity:) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z \in R$.

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(1) is trivial.
(2) is also very easy (it is a standard exercise in the first lecture of every course of Lie algebras; 12 products of x, y, z, in all their possible 6 orders, 6 with plus and 6 with minus, of the form $(xy)z - x(yz)$ say, and they pairwise cancel because the operation $\cdot$ in the ring R is associative.

Associativity of $\cdot$ is not really necessary to prove the Jacobi identity, something less is sufficient.

Algebras

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The *opposite* M^{op} of an algebra M is defined taking as multiplication in M^{op} the mapping $(x, y) \mapsto yx$.

Homomorphisms.

If M and M' are two k-algebras, a k-linear mapping $\varphi\colon M \to M'$ is a *k-algebra homomorphism* if $\varphi(xy) = \varphi(x)\varphi(y)$ for every $x, y \in M$.

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k-algebras form a variety in the sense of Universal Algebra.

If M is any k-algebra, its endomorphisms form a monoid, that is, a semigroup with a two-sided identity, with respect to composition $\circ$ of mappings.

Pre-Lie algebras

A *pre-Lie k-algebra* is a k-algebra $(M, \cdot)$ satisfying the identity

$$(x \cdot y) \cdot z - x \cdot (y \cdot z) = (y \cdot x) \cdot z - y \cdot (x \cdot z) \tag{1}$$

for every $x, y, z \in M$.

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For any k-algebra $(M, \cdot)$, defining the commutator $[x, y] = x \cdot y - y \cdot x$ for every $x, y \in M$, the algebra $(M, [-, -])$ is anticommutative (i.e., $[x, y] = -[y, x]$ and $[x, x] = 0$), but it is not-necessarily a Lie algebra.

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If $(M, \cdot)$ is a pre-Lie algebra, one gets that $(M, [-, -])$ is a Lie algebra, called the *Lie algebra sub-adjacent* to the pre-Lie algebra $(M, \cdot)$.

Pre-Lie algebras

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It is easily seen that the category of left-symmetric algebras and the category of right-symmetric algebras are isomorphic (the categorical isomorphism is given by $M \mapsto M^{\text{op}}$).

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A much better term would have been “pre-associative algebras”.

A hierarchy of algebras

(1) Associative algebras.

(2) Pre-Lie algebras.

(3) Lie admissible algebras (= algebras $(M, \cdot)$ for which $(M, [-,-])$ is a Lie algebra).

(4) (Arbitary non-associative) algebras.

Associator. Lie admissible algebras

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Being a Lie-admissible algebra is equivalent to

$$(x, y, z) + (y, z, x) + (z, x, y) = (y, x, z) + (x, z, y) + (z, y, x)$$

for every $x, y, z \in M$.

Derivations on $k[x_1, \ldots, x_n]^n$.

Let k be a commutative ring with identity, $n \geq 1$ be an integer, and $k[x_1, \ldots, x_n]$ be the ring of polynomials in the n indeterminates $x_1, \ldots, x_n$ with coefficients in k. Let A be the free $k[x_1, \ldots, x_n]$-module $k[x_1, \ldots, x_n]^n$ with free set $\{e_1, \ldots, e_n\}$ of generators. As a k-module, A is the free k-module with free set of generators the set $\{\, x_1^{i_1} \ldots x_n^{i_n} e_j \mid i_1, \ldots, i_n \geq 0,\ j = 1, \ldots, n\}$.

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Define a multiplication on A setting, for every $u = (u_1, \dots, u_n), v = (v_1, \dots, v_n) \in\ A$,

$$v \cdot u = (\sum_{j=1}^{n} v_j \frac{\partial u_1}{\partial x_j}, \dots, \sum_{j=1}^{n} v_j \frac{\partial u_n}{\partial x_j}).$$

Then A is a pre-Lie k-algebra.

Rooted trees.

Recall that a *tree* is an undirected graph in which any two vertices are connected by exactly one path, or equivalently a connected acyclic undirected graph. A *rooted tree* of degree n is a pair (T, r), where T is a tree with n vertices, and its *root* r is a vertex of T.

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Let k be a commutative ring with identity and $\mathcal{T}_n$ be the free k-module with free set of generators the set of all isomorphism classes of rooted trees of degree n. Set

$$\mathcal{T} := \bigoplus_{n \geq 1} \mathcal{T}_n.$$

Rooted trees.

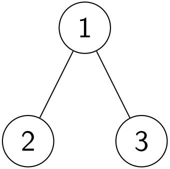
Define a multiplication on $\mathcal{T}$ setting, for every pair T_1, T_2 of rooted trees,

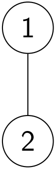
$$T_1 \cdot T_2 = \sum_{v \in V(T_2)} T_1 \circ_v T_2,$$

where $V(T_2)$ is the set of vertices of T_2, and $T_1 \circ_v T_2$ is the rooted tree obtained by adding to the disjoint union of T_1 and T_2 a further new edge joining the root vertex of T_1 with the vertex v of T_2. The root of $T_1 \circ_v T_2$ is defined to be the same as the root of T_2. To get a multiplication on $\mathcal{T}$, extend this multiplication by k-bilinearity.

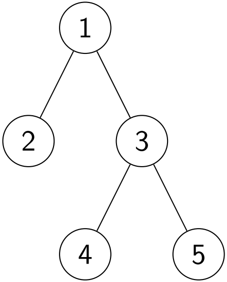
Rooted trees.

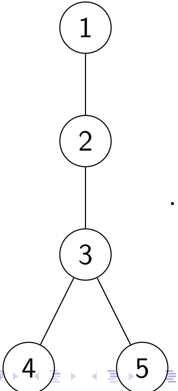
Let us give an example. Suppose

$T_1 =$ 

and $T_2 =$ 

Then

$T_1 \circ_1 T_2 =$ 

and $T_1 \circ_2 T_2 =$ 

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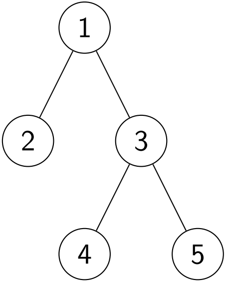
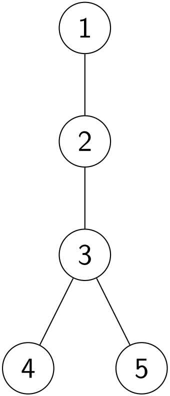
Rooted trees.

Therefore

$T_1 \cdot T_2 =$  $+$ 

In this way, one gets a pre-Lie k-algebra $\mathcal{T}$.

Pre-morphisms

A k-module morphism $\varphi\colon M \to M'$, where M, M' are arbitrary (not-necessarily associative) k-algebras, is a *pre-morphism* if $\varphi(xy) - \varphi(x)\varphi(y) = \varphi(yx) - \varphi(y)\varphi(x)$ for every $x, y \in M$.

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Lemma

(a) *Every k-algebra morphism is a pre-morphism.*
(b) *The composite mapping of two pre-morphisms is a pre-morphism.*
(c) *The inverse mapping of a bijective pre-morphism is a pre-morphism.*

Pre-morphisms

For any (not-necessarily associative) k-algebra M, there is a mapping $\lambda\colon M \to \mathrm{End}({}_k M)$, where $\lambda\colon x \mapsto \lambda_x$, $\lambda_x\colon M \to M$, and $\lambda_x(a) = xa$.

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(1) The mapping λ is a k-algebra morphism if and only if M is associative.

(2) The mapping λ is a pre-morphism if and only if M is a pre-Lie algebra.

Two categories

Category Alg_k of (not-necessarily associative) k-algebras with their homomorphisms.

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There is a functor $U\colon \mathsf{Alg}_{k,p} \to \mathsf{Alg}_k$ that associates with any k-algebra $(A, \cdot)$ its sub-adjacent anticommutative algebra $(A, [-,-])$, where $[x, y] = xy - yx$ for every $x, y \in A$. It associates with any pre-morphism $f\colon (A, \cdot) \to (B, \cdot)$ in Alg_k, the same mapping $U(f) = f\colon (A, [-,-]) \to (B, [-,-])$.

Examples

(1) The center $Z(A)$ of an associative algebra M is $Z(M) = \{ x \in M \mid [x, M] = \{0\} \}$. It is a pre-ideal of M.

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(2) The kernel of any pre-morphism (=the inverse image of 0) is always a pre-ideal.

Pre-derivations

Corresponding to the notion of pre-morphism, there is a notion of pre-derivation. We say that a k-module endomorphism $\delta\colon M \to M$, where M is an arbitrary (not-necessarily associative) k-algebra, is a *pre-derivation* if

$$\delta(xy) - \delta(x)y - x\delta(y) = \delta(yx) - \delta(y)x - y\delta(x)$$

for every $x, y \in M$.

Modules over a pre-Lie algebra

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There is a natural notion of module over a pre-Lie algebra:

A module M over a pre-Lie k-algebra $(A, \cdot)$ is a k-module M with a pre-morphism $\lambda\colon (A, \cdot) \to (\mathrm{End}({}_k M), \circ)$.

There is a clear relation between the notions introduced until now in this paper and the concepts of Lie derivation and Jordan derivation for an associative algebra. Lie derivations and Jordan derivations for associative algebras were introduced and studied by Ancochea (1948), Jacobson (1937), Herstein (1957-1961), Brešar (1993-2005), and several other mathematicians in the past decades.

A Lie derivation of an associative algebra $(M, \cdot)$ is a mapping $(M, \cdot) \to (M, \cdot)$ that is a derivation $(M, [-,-]) \to (M, [-,-])$ of the Lie algebra $(M, [-,-])$.

Similarly for Jordan derivations of an associative algebra $(M, \cdot)$, where the Lie algebra $(M, [-,-])$ is replaced by the Jordan algebra $(M, \circ)$ in the definition. Thus, for an associative algebra, our pre-derivations are exactly Lie derivations.

Similarly, consider the notion of generalized derivation as it was introduced in Brešar (1991). In that paper, a *generalized derivation* $f\colon (M,\cdot) \to (M,\cdot)$ of an associative k-algebra $(M,\cdot)$ is a k-module morphism for which there exists a derivation $d\colon M \to M$ such that $f(xy) = f(x)y + xd(y)$ for every $x, y \in M$. This is equivalent to the existence of a k-module morphism $d\colon M \to M$ for which

$$\begin{cases} f(xy) = f(x)y + xd(y) \\ d(xy) = d(x)y + xd(y) \end{cases}$$

for every $x, y \in M$. Equivalently, if and only if there exists a derivation $d\colon M \to M$ such that $f(x)y - f(xy) = d(x)y - d(xy)$ for every $x, y \in M$. This equation is very similar to other equations in this paper, in the sense that the expression $f(x)y - f(xy)$ can be also seen as a sort of associator (f, x, y).

If we want to be also bolder, we can say that, for any k-algebra M, the k-algebra $\mathrm{End}({}_kM)$ of all endomorphisms of the k-module ${}_kM$ is an associative k-algebra, and M is an $\mathrm{End}({}_kM)$-$\mathrm{End}({}_kM)$-bimodule over this associative k-algebra if we set $f \cdot x = f(x)$ and $x \cdot f = 0$ for every $x \in M$ and $f \in \mathrm{End}({}_kM)$. Then a k-module morphism $f \in \mathrm{End}({}_kM)$ is:

(1) a right M-module morphism if and only if $(f, x, y) = 0$ for every $x, y \in M$ (same definition as for an associative algebra),

(2) a derivation if and only if $(f, x, y) = (x, f, y)$ for every $x, y \in M$ (same definition as for a pre-Lie algebra), and

(3) a generalized derivation if and only if there exists a derivation d of M for which $(f, x, y) = (d, x, y)$ for every $x, y \in M$.

We prefer not to use the terminology Lie homomorphism, Jordan homomorphism, Lie derivation, Lie ideal, and Lie subalgebra as in Herstein (1991) for the simple reason that if $(M, \cdot)$ is a non-associative algebra, then $(M, [-,-])$ and $(M, \circ)$ are not a Lie algebra and a Jordan algebra respectively, but only an anticommutative algebra and a commutative algebra respectively.

Correspondingly, we prefer to use the terms pre-Lie-morphism, generalized morphism, pre-derivation, pre-ideal, and pre-subalgebra, though we understand the possible problematic with this terminology.

Commutator of two ideals in a pre-Lie algebra

For pre-Lie algebras, Smith=Huq.

Theorem

The commutator $[I, J]$ *of two ideals* I *and* J *of a pre-Lie algebra* A *is the ideal of* A *generated by the subset* $\{\, i \cdot j, j \cdot i \mid i \in I,\ j \in J \,\}$.

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Conversely, for any morphism $\varphi \colon k \to A$ the corresponding idempotent element of A is $\varphi(1)$.

Dorroh extension of a pre-Lie algebra

For any fixed pre-Lie k-algebra A it is possible to construct the k-module direct sum $A \oplus k$ with multiplication defined by

$$(x, \alpha)(y, \beta) = (x \cdot y + \beta x + \alpha y, \alpha\beta)$$

for every $(x, \alpha), (y, \beta) \in A \oplus k$.

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(This k-algebra $A \oplus k$, usually denoted $A\#k$, is a particular case of semidirect product of pre-Lie algebras.).

Dorroh extension of a pre-Lie algebra

Let $\mathrm{PreL}_{k,1}$ be the category of all unital pre-Lie k-algebras. Its objects are the pre-Lie k-algebras A with an identity. Its morphisms $f\colon A \to B$ are the k-algebra morphisms f such that $f(1_A) = 1_B$.

Dorroh extension of a pre-Lie algebra

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Its objects are all the pairs (A, ε_A), where A is a unital pre-Lie k-algebra and $\varepsilon_A \colon A \to k$ is a morphism in $\mathrm{PreL}_{k,1}$ that is a left inverse for $\varphi_{1_A} \colon k \to A$, $\varphi_{1_A} \colon \lambda \in k \to \lambda \cdot 1_A$:

$$k \xrightarrow{\varphi_{1_A}} A \xrightarrow{\varepsilon_A} k.$$

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$$k \xrightarrow{\varphi_{1_A}} A \xrightarrow{\varepsilon_A} k.$$

The morphisms $f \colon (A, \varepsilon_A) \to (B, \varepsilon_B)$ are the morphisms $f \colon A \to B$ in $\mathrm{PreL}_{k,1}$ such that $\varepsilon_B f = \varepsilon_A$. For instance, the k-algebra $A\#k$ is clearly a unital k-algebra with augmentation: the augmentation is the canonical projection $\pi_2 \colon A\#k = A \oplus k \to k$ onto the second summand.

Dorroh extension of a pre-Lie algebra

It is easy to see that:

Theorem

There is a category equivalence $F\colon \mathrm{PreL}_k \to \mathrm{PreL}_{k,1,a}$ *that associates with any object* A *of* PreL_k *the* k*-algebra with augmentation* $F(A) := (A\#k, \pi_2)$*. The quasi-inverse of* F *is the functor* $\mathrm{PreL}_{k,1,a} \to \mathrm{PreL}_{k,}$ *that associates with each unital pre-Lie* k*-algebra with augmentation* (A, ε_A) *the kernel* $\ker(\varepsilon_A)$ *of the augmentation.*

Idempotent endomorphisms

Proposition

Let M be a k-algebra. There is a bijection betwee the set $E := \{\, e \in \mathrm{End}_k(M) \mid e$ is a morphism and $e\colon M \to M$ is idempotent$\,\}$ of all idempotent endomorphisms of M and the set P of all pairs (K, B), where K is a ideal of M, B is a k-subalgebra of B, and ${}_kM = K \oplus B$ as a k-module.

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Conversely, the idempotent endomorphism that corresponds to a pair $(K, B) \in P$ is the composite mapping of the second canonical projection $\pi_2\colon {}_kM = K \oplus B \to B$ and the inclusion $\varepsilon_2\colon B \hookrightarrow {}_kM$.

If M is a k-algebra, K is a ideal of M, B is a k-subalgebra of B, and ${}_kM = K \oplus B$ as a k-module, there there is a pair (λ, ρ) of k-linear mappings $B \to \mathrm{End}(I_k)$ defined by $\lambda(b)(i) = bi$ and $\rho(b)(i) = ib$ for every $b \in B$ and $i \in I$.

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In the particular case where M is a pre-Lie k-algebra, one finds that:
(a) $\lambda\colon (B, \cdot) \to (\mathrm{End}(I_k), \circ)$ is a pre-morphism.
(b) $\rho_a \circ \lambda_b - \lambda_b \circ \rho_a = \rho_a \circ \rho_b - \rho_{b \cdot a}$ for every $a, b \in B$.
(c) $\lambda_a(i) \cdot j - \lambda_a(i \cdot j) = \rho_a(i) \cdot j - i \cdot \lambda_a(j)$ for every $a \in B$ and $i, j \in I$.
(d) $\rho_a(i \cdot j) - i \cdot \rho_a(j) = \rho_a(j \cdot i) - j \cdot \rho_a(i)$ for every $a \in B$ and $i, j \in I$.

Action of a pre-Lie algebra on another pre-Lie algebra

Let I and B be pre-Lie k-algebras and (λ, ρ) a pair of k-linear mappings $B \to \mathrm{End}(I_k)$ such that:
(a) $\lambda\colon (B, \cdot) \to (\mathrm{End}(I_k), \circ)$ is a pre-morphism.
(b) $\rho_a \circ \lambda_b - \lambda_b \circ \rho_a = \rho_a \circ \rho_b - \rho_{b \cdot a}$ for every $a, b \in B$.
(c) $\lambda_a(i) \cdot j - \lambda_a(i \cdot j) = \rho_a(i) \cdot j - i \cdot \lambda_a(j)$ for every $a \in B$ and $i, j \in I$.
(d) $\rho_a(i \cdot j) - i \cdot \rho_a(j) = \rho_a(j \cdot i) - j \cdot \rho_a(i)$ for every $a \in B$ and $i, j \in I$.
On the k-module direct sum $I \oplus B$ define a multiplication $*$ setting

$$(i, b) * (j, c) = (i \cdot j + \lambda_b(j) + \rho_c(i), b \cdot c)$$

for every $(i, b), (j, c) \in I \oplus B$. Then $(I \oplus B, *)$ is a pre-Lie k-algebra (the semidirect product).

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Here by idempotent pre-endomorphism $e\colon M \to M$ of a k-algebra M we mean a k-linear mapping such that $e^2 = e$ and

$$e(xy) - e(x)e(y) = e(yx) - e(y)e(x) \tag{2}$$

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Recall that it is possible to associate to any k-algebra $(M, \cdot)$ the anticommutative k-algebra $(M, [-, -])$.

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Dualizing. Two possible ways

Our definition of pre-morphism was
$\varphi(xy) - \varphi(x)\varphi(y) = \varphi(yx) - \varphi(y)\varphi(x)$ for every $x, y \in M$.

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The first is replacing our condition with

$$\varphi(xy) - \varphi(x)\varphi(y) = -(\varphi(yx) - \varphi(y)\varphi(x))$$

and the second is replacing it with

$$\varphi(xy) + \varphi(x)\varphi(y) = \varphi(yx) + \varphi(y)\varphi(x).$$

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The first possibility leads to the notion of Jordan algebras, the second one to anti-pre-Lie algebras.

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Jordan algebra = k-algebra for which
$xy = yx$ (commutative algebra)
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In a Jordan algebra powers x^n of an element work well:
(1) $x^n = x \cdots x$ is independent of how we parenthesize the expression on the right.
(2) $\lambda_{x^m} \circ \lambda_{x^n} = \lambda_{x^n} \circ \lambda_{x^m}$ for every pair of integers $m, n \geq 0$.

Anti-pre-Lie algebras

This is an extremely recent notion [Guilai Liu and Chengming Bai, *Anti-pre-Lie algebras, Novikov algebras and commutative 2-cocycles on Lie algebras*, arXiv https://doi.org/10.48550/arXiv.2207.06200].

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The k-algebra A is an *anti-pre-Lie k-algebra* if

$$(x \cdot y) \cdot z + x \cdot (y \cdot z) = (y \cdot x) \cdot z + y \cdot (x \cdot z) \qquad (3)$$

and

$$[x, y] \cdot z + [y, z] \cdot x + [z, x] \cdot y = 0 \qquad (4)$$

for every $x, y, z \in A$.