Jaffard families and extension of star operations

Dario Spirito

Università di Roma Tre

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Star operations

- We will always take an integral domain D with quotient field K.
- Let $\mathbf{F}(D)$ be the set of D-submodules of K.
- Let $\mathcal{F}(D)$ be the set of *fractional ideals* of D, i.e., of the $I \in \mathbf{F}(D)$ such that $xI \subseteq D$ for some $x \in K$.

Definition

A star operation on D is a map $* : \mathcal{F}(D) \longrightarrow \mathcal{F}(D)$ such that

- $I \subseteq I^*$;
- $I \subseteq J \Longrightarrow I^* \subseteq J^*$;
- $(I^*)^* = I^*$.
- $(xI)^* = x \cdot I^*$;
- $D^* = D$.

Examples

- The identity $d : I \mapsto I$.
- The v-operation $v : I \mapsto (D : (D : I))$.
- The t-operation:

$$t : I \mapsto \bigcup \{J^v \mid J \subseteq I, J \text{ is finitely generated}\}.$$

- If $\mathcal{Y} \subseteq \mathrm{Over}(D)$ and $\bigcap_{T \in \mathcal{Y}} T = D$, we can define

$$\wedge_{\mathcal{Y}} : I \mapsto \bigcap_{T \in \mathcal{Y}} IT.$$

- If $\Delta \subseteq \mathrm{Spec}(D)$ and $\bigcap_{P \in \Delta} D_P = D$, we can define

$$s_\Delta : I \mapsto \bigcap_{P \in \Delta} ID_P.$$

Types of closure operations

- $*$ is of finite type if, for every I,

$$I^* = \bigcup\{F^* \mid F \subseteq I,\ F \text{ is finitely generated}\}$$

- $*$ is spectral if it is in the form s_Δ.
- $*$ is stable if $(I \cap J)^* = I^* \cap J^*$ for all I, J.
- $*$ is Noetherian if the set

$$\mathcal{I}^*(D) := \{I \in \mathcal{F}(D) \mid I \subseteq D,\ I = I^*\}$$

satisfies the ascending chain condition.

Structures on $\mathrm{Star}(D)$

- Order structure: $*_1 \leq *_2$ if $I^{*_1} \subseteq I^{*_2}$ for every $I \in \mathcal{F}(D)$.
  - $\mathrm{Star}(D)$ is a complete lattice.
  - v is the maximum of $\mathrm{Star}(D)$.
  - t is the maximum of $\mathrm{Star}_f(D)$.
- Topological structure: the topology is the one generated by the sets

$$V_I := \{* \in \mathrm{Star}(D) \mid 1 \in I^*\}.$$

  - $\mathrm{Star}_f(D)$ is better behaved than $\mathrm{Star}(D)$.
- Set structure: study of the cardinality.
  - For example, when is $\mathrm{Star}(D)$ finite? When $|\mathrm{Star}(D)| = 1$?
  - There are no general results, but some can be said when D is Noetherian or when it is integrally closed.
  - For example, if D is Noetherian then $|\mathrm{Star}(D)| = 1$ if and only if D is Gorenstein of dimension 1.

Extensions of star operations

Let D be an integral domain and T a flat overring of D.

Definition

A star operation $*$ on D is extendable to T if the map

$$\begin{aligned} *_T \colon \mathcal{F}(T) &\longrightarrow \mathcal{F}(T) \\ IT &\longmapsto I^*T \end{aligned}$$

is well-defined.

- Equivalently, $*$ is extendable if $IT = JT$ implies $I^*T = J^*T$.
- Since T is flat, every ideal of T is an extension of an ideal of D.
- Not every star operation is extendable.
- Finite-type operations are extendable.
- If $*$ is of finite type (respectively, spectral, Noetherian) then so is $*_T$.

Extension as a map

- Let $\mathrm{ExtStar}(D;T)$ be the set of star operations of D that are extendable to T.
- Extension defines a map

$$\begin{aligned}\lambda_{D,T}\colon \mathrm{ExtStar}(D;T) &\longrightarrow \mathrm{Star}(T)\\ * &\longmapsto *_T.\end{aligned}$$

- $\lambda_{D,T}$ is continuous.
- $\lambda_{D,T}$ is surjective if and only if its image contains the v-operation (on T).
- $\lambda_{D,T}$ is almost never injective.

Restriction of star operations

- The concept dual to extension is restriction: if $* \in \mathrm{Star}(T)$, its restriction to D is
$$*\wedge v : I \mapsto (IT)^* \cap I^v.$$
- We can see restriction as a map
$$\begin{aligned}\rho_{T,D} \colon \mathrm{Star}(T) &\longrightarrow \mathrm{Star}(D)\\ * &\longmapsto *\wedge v\end{aligned}$$
- $\rho_{T,D}$ is continuous.
- Restriction doesn't preserve properties (unless v has them).
- $\rho_{T,D}$ is almost never surjective.

Families of overrings

- It is more useful to work with *families* of overrings:

$$\begin{aligned}\lambda_\Theta\colon \mathrm{ExtStar}(D;\Theta) &\longrightarrow \prod_{T\in\Theta}\mathrm{Star}(T)\\ * &\longmapsto (*_T)_{T\in\Theta}\end{aligned}$$

  where $\mathrm{ExtStar}(D;\Theta) := \bigcap_{T\in\Theta}\mathrm{ExtStar}(D;T)$.
- In the same way, we can define

$$\begin{aligned}\rho_\Theta\colon \prod_{T\in\Theta}\mathrm{Star}(T) &\longrightarrow \mathrm{Star}(D)\\ (*^{(T)})_{T\in\Theta} &\longmapsto \inf\{\rho_T(*^{(T)}) \mid T\in\Theta\}.\end{aligned}$$

λ_Θ and ρ_Θ

- λ_Θ is continuous.
- If Θ is locally finite, ρ_Θ is continuous.
    - Θ is *locally finite* (or *of finite character*) if, for every $x \in K$, there are only finitely many $T \in \Theta$ such that $xT \subsetneq T$.
- If Θ is complete, then λ_Θ is injective; if Θ is also locally finite, then λ_Θ is a topological embedding.
    - Θ is *complete* if $I = \bigcap_{T\in\Theta} IT$ for all $I \in \mathcal{F}(D)$.
- Problems:
    - What is $\mathrm{ExtStar}(D;\Theta)$?
    - Is λ_Θ surjective?
- With some hypothesis, we can solve these problems:
    - Suppose D is Noetherian, integrally closed, locally finite, and that $\dim(D) = 2$: then,

$$\mathrm{Star}(D) \simeq \prod_{M\in\mathrm{Max}(D)} \mathrm{Star}(D_M).$$

    - It is possible to weaken "integrally closed", but not the other hypothesis.

Jaffard families

Definition

A Jaffard family of D is a family $\Theta \subseteq \mathrm{Over}(D)$ of flat overrings such that:

- *Θ is complete;*
- *Θ is locally finite;*
- *$TS = K$ for every $T, S \in \Theta$, $T \neq S$ (Θ is independent).*

- The second and the third condition are equivalent to $T \cdot \Theta^{\perp}(T) = K$, where $\Theta^{\perp}(T)$ is the intersection of all $S \in \Theta \setminus T$.
- In particular, if $P \in \mathrm{Spec}(D)$, $P \neq (0)$, then there is exactly one $T \in \Theta$ such that $PT \neq T$.
- Jaffard families are in bijective correspondence with particular partitions of $\mathrm{Max}(D)$ (*Matlis partitions*).

Jaffard families (2)

- Jaffard families generalize the concept of h-local domain: indeed, the set $\{D_M \mid M \in \mathrm{Max}(D)\}$ is a Jaffard family if and only if D is h-local.
  - A domain is h-local if it is locally finite and every prime is contained in only one maximal ideal.
- If $\{X_\alpha\}_{\alpha\in A} \subseteq \mathbf{F}(D)$ and $\bigcap_\alpha X_\alpha \neq (0)$, and $T \in \Theta$, then

$$\left(\bigcap_{\alpha\in A} X_\alpha\right) T = \bigcap_{\alpha\in A} X_\alpha T.$$

- If M is a torsion D-module, then

$$M \simeq \bigoplus_{T\in\Theta} M \otimes_D T.$$

  - In particular, if $I \neq (0)$ is an ideal of D, then

$$\frac{D}{I} \simeq \bigoplus_{T\in\Theta} \frac{T}{IT}.$$

The main theorem

Theorem

Let D be an integral domain and Θ be a Jaffard family of D. Then:

- *every* $* \in \mathrm{Star}(D)$ *is extendable to every* $T \in \Theta$*;*
- λ_Θ *and* ρ_Θ *are homeomorphisms between* $\mathrm{Star}(D)$ *and* $\displaystyle\prod_{T \in \Theta} \mathrm{Star}(T)$.

- The same holds if, instead of the set of all star operations, we consider only finite-type, spectral, stable, or Noetherian star operations.
- The theorem does *not* hold for semistar operations.

Consequences of the main theorem

- If $\dim(D) = 1$ and D is locally finite, then

$$\mathrm{Star}(D) \simeq \prod_{M \in \mathrm{Max}(D)} \mathrm{Star}(D_M).$$

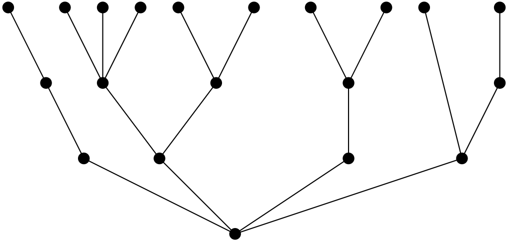
- D has an m-canonical ideal if and only if:
  - D is h-local;
  - D_M has an m-canonical ideal for every $M \in \mathrm{Max}(D)$;
  - $|\mathrm{Star}(D_M)| > 1$ for only finitely many $M \in \mathrm{Max}(D)$.
- If $* \in \mathrm{Star}(D)$, then

$$\frac{\mathrm{Cl}^*(D)}{\mathrm{Pic}(D)} \simeq \bigoplus_{T \in \Theta} \frac{\mathrm{Cl}^{*_T}(T)}{\mathrm{Pic}(T)}.$$

Prüfer domains

When D is a Prüfer domain, there is a natural candidate for Θ.

- Say $M, N \in \mathrm{Max}(D)$ are *dependent* if there is a prime ideal $P \neq (0)$ such that $P \subseteq M \cap N$.
- Dependence is an equivalence relation.
- For all $M \in \mathrm{Max}(D)$, define $T(M)$ as the intersection of D_N, as N ranges in the equivalence class of M.
- Take $\Theta := \{T(M) \mid M \in \mathrm{Max}(D)\}$.



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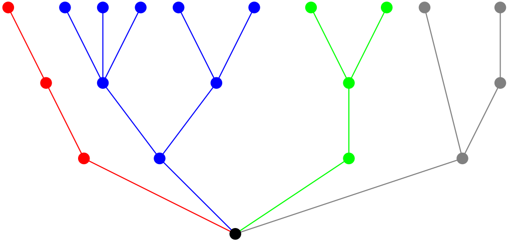


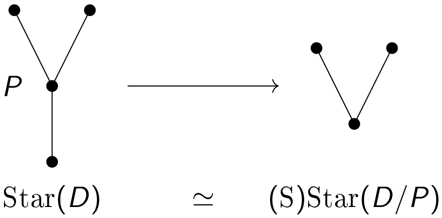
Prüfer domains (2)

- Problem: Θ may not be a Jaffard family.
  - For example, it may not be locally finite (e.g., an almost Dedekind domain which is not Dedekind).
  - It works if we restrict to locally finite domains.
- Problem: if T is not a valuation domain, we don't know $\mathrm{Star}(T)$.
  - Suppose D is semilocal, or locally finite and finite-dimensional.
  - Then, $\mathrm{Jac}(D)$ contains a prime ideal P.
  - We want to link $\mathrm{Star}(T)$ and $\mathrm{Star}(T/P)$.

Cutting the branch

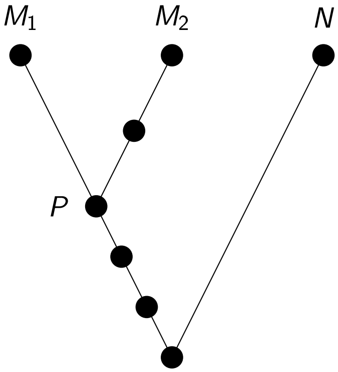
Suppose there is a prime ideal $P \neq (0)$ inside the Jacobson radical.

- $P = PD_P$ (P is *divided*).
- Every D-submodule of K is a fractional ideal.
- Non-divisorial ideals correspond to D/P-submodules of D_P/P.
- [Fontana and Park, 2004; Houston, Mimouni and Park, 2014] Star operations correspond to semistar operations such that $(D/P)^* = (D/P)$.



An inductive argument

Suppose we can go from $(\mathrm{S})\mathrm{Star}(R)$ to $\mathrm{Star}(R)$, for every (semilocal) Prüfer domain.



$\mathrm{Star}(D)$

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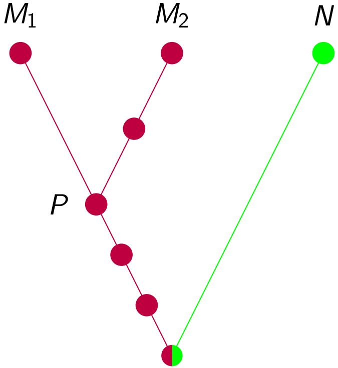

$$\begin{tikzcd} \mathrm{Star}(D) & \\ \mathrm{Star}(D_1) \arrow[u, "\text{extension}"] & \mathrm{Star}(D_N) \arrow[ul] \end{tikzcd}$$

$$D_1 := D_{M_1} \cap D_{M_2}$$

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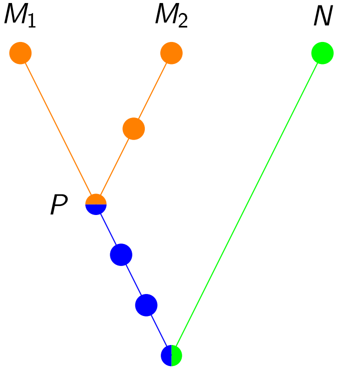

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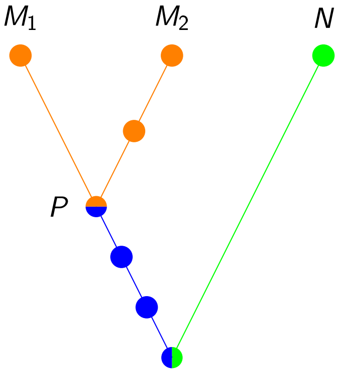

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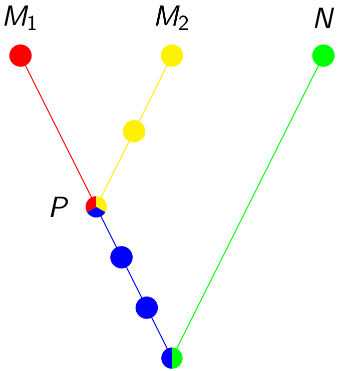

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Branching points

- Up to the passage $\mathrm{Star}(R) \to (\mathrm{S})\mathrm{Star}(R)$, the algorithms needs two things:
- the geometry of Spec(D): we need to know the "branching points" of the spectrum to know the places in which we cut and in which we localize;
- the star operations on D_Q/PD_Q, where $P \subsetneq Q$ are successive branching points of D.
  - Every D_Q/PD_Q is a valuation domain.
  - $\mathrm{Star}(D_Q/PD_Q)$ depends only on whether Q is idempotent or not.

Fractional star operations

Star and semistar operations are defined similarly: closure operations on a set of modules satisfying $(xI)^* = x \cdot I^*$.

	$D = D^*$	D^* arbitrary
$\mathcal{F}(D)$	star operations	
$\mathbf{F}(D)$		semistar operations

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	$D = D^*$	D^* arbitrary
$\mathcal{F}(D)$	star operations	fractional star operations
$\mathbf{F}(D)$	(semi)star operations	semistar operations

- Fractional star operations satisfy the main theorem.
- We can control the passage from $\mathrm{FStar}(D)$ to $\mathrm{SStar}(D/P)$.

Gluing fractional star operations

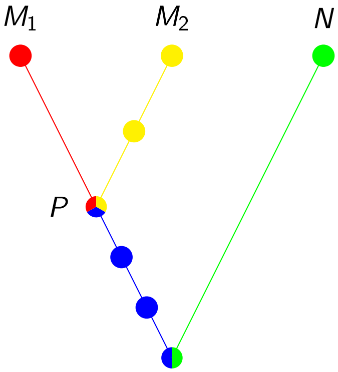
Let D be a semilocal Prüfer domain and let Θ be its Jaffard family.

- There is a (explicit, unique) family $\mathrm{SkOver}(D)$ of overrings such that every D-submodule of K is a fractional ideal over exactly one $U \in \mathrm{SkOver}(D)$.
- $* \in \mathrm{SStar}(D)$ is determined by:
  - the set Δ_* of $U \in \mathrm{SkOver}(D)$ such that $U^* \in \mathcal{F}(U)$;
  - $*|_{\mathcal{F}(U)} \in \mathrm{FStar}(U)$, for $U \in \Delta_*$.
- The set of $*$ such that $\Delta_* = \Delta$ is empty or isomorphic to

$$\prod_{T \in \Theta} \mathrm{hom}(\Delta(T), \mathrm{FStar}(T))$$

  where $\Delta(T) := \{U \in \mathrm{SkOver}(D) \mid U \subseteq T\}$ and hom are the order-preserving maps.

An inductive argument (2)



$$\begin{tikzcd}
\mathrm{SStar}(D) & \\
\mathrm{FStar}(D) \arrow[u, "\text{gluing}"] & \\
\mathrm{FStar}(D_1) \arrow[u, "\text{extension}"] & \mathrm{FStar}(D_N) \arrow[ul] \\
\mathrm{SStar}(D_1/P) \arrow[u, "\text{cutting}"] & \mathrm{FStar}(D_P) \arrow[ul] \\
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\end{tikzcd}$$

$$D_1 := D_{M_1} \cap D_{M_2}$$

Prüfer domains with the same star operations

- For $\mathrm{(S)Star}(D)$ and $\mathrm{Star}(D)$ the reasoning is similar.
  - Even for them, you still have to use $\mathrm{FStar}(D)$.
- Let D, D' be semilocal Prüfer domains such that:
  - there is an isomorphism ϕ between the set of the branching points of D and D';
  - $\mathrm{SStar}(D_P) \simeq \mathrm{SStar}(D'_{\phi(P)})$ for all branching points P.

  Then, $\mathrm{SStar}(D) \simeq \mathrm{SStar}(D')$.
  - Suppose M is idempotent if and only if $\phi(M)$ is idempotent, for every maximal ideal M. Then, $\mathrm{Star}(D) \simeq \mathrm{Star}(D')$.
- Let D, D' be semilocal Prüfer domains such that:
  - there is a homeomorphism $\phi : \mathrm{Spec}(D) \longrightarrow \mathrm{Spec}(D')$;
  - P is idempotent if and only if $\phi(P)$ is idempotent.

  Then, $\mathrm{SStar}(D) \simeq \mathrm{SStar}(D')$ and $\mathrm{Star}(D) \simeq \mathrm{Star}(D')$.

Star operations on Prüfer domains

Suppose D is a semilocal Prüfer domain.

- If $\dim(D)$ is finite, so are $\mathrm{Star}(D)$ and $\mathrm{SStar}(D)$.
  - You can actually calculate the cardinality.
- The set of stable star operations is isomorphic to $\prod_{M\in\mathrm{Max}(D)} \mathrm{Star}(D_M)$.
  - Equivalently, to the power set of $\{M \in \mathrm{Max}(D) \mid M \neq M^v\}$.
  - D is h-local if and only if every star operation is stable.
- If $I = I^*$ and $J = J^*$ are $*$-invertible, then so is $I + J$.
  - L is $*$-invertible if $(L(D : L))^* = D$.
- For every $* \in \mathrm{Star}(D)$,

$$\mathrm{Cl}^*(D) \simeq \bigoplus_{\substack{M\in\mathrm{Max}(D)\\ M\neq M^*}} \mathrm{Cl}^v(D_M)$$

  and we know the right hand side: it is (0) or $\mathbb{R}/H$ for some subgroup H depending on the value group of D_M.

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