‘Every’ set is a set of lengths of some numerical monoid

W.A. Schmid

LAGA, Université Paris 8, France

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Numerical monoids

A numerical monoid H is a submonoid of $(\mathbb{N}_0, +)$ with finite complement.

A numerical monoid is finitely generated and reduced. Let $\mathcal{A}(H) = \{n_1 < \cdots < n_t\}$ denote the set of irreducible elements.

(Every nontrivial submonoid of $(\mathbb{N}_0, +)$ is isomorphic to a numerical monoid.)

Since

$$\underbrace{n_1 + \cdots + n_1}_{n_t} = \underbrace{n_t + \cdots + n_t}_{n_1}$$

it is follows that H is not a unique factorization monoid unless $t = 1$ (which implies $n_1 = 1$ and $H = \mathbb{N}_0$).

The question arises to understand the arithmetic of these monoids more precisely.

Sets of lengths

A monoid $(H, +)$ (commutative, cancellative) is called

1. *atomic* if each non-zero element *a* is the sum (of finitely many) irreducible elements.
2. *factorial* if there is an essentially unique factorization into irreducibles (i.e., up to ordering and associates).

Sets of lengths, II

If

$$a = a_1 + \cdots + a_n$$

with irrededucibles a_i, then n is called a length of a.

$$\mathsf{L}(a) = \{n \colon n \text{ is a length }\}.$$

For $a = 0$ we set $\mathsf{L}(a) = \{0\}$.
The *system of sets of lengths* is

$$\mathcal{L}(H) = \{\mathsf{L}(a) \colon a \in H\}.$$

Sets of lengths, III

In general, sets of lengths can be infinite. Yet, for finitely generated monoids (as well as many other classes of interest) they are *finite*.
The property is called BF (bounded factorization).
Note: Each set of lengths is finite, but still in general there are infinitely many sets.
So

$$\mathcal{L}(H) \subset \mathbb{P}_{\text{fin}}(\mathbb{N}_0).$$

General properties of systems of sets of lengths (of BF)

Let $L, L' \in \mathcal{L}(H)$.

- If $0 \in L$, then $L = \{0\}$.
- If $1 \in L$, then $L = \{1\}$.
- Let $S = L + L' = \{l + l' \colon l \in L,\ l' \in L'\}$. There exists some $L'' \in \mathcal{L}(H)$ such that $S \subset L''$.

If $\mathcal{L}(H)$ contains some L with $|L| \geq 2$, then $\mathcal{L}(H)$ contains arbitrarily large sets.
Moreover

$$\mathcal{L}(H) \subset \{\{0\}, \{1\}\} \cup \mathbb{P}_{\text{fin}}(\mathbb{N}_{\geq 2}).$$

Dichotomy (for BF-structures)

- Either $|\mathsf{L}(a)| = 1$ for each a,
- or for each n there exists a a_n such that $|\mathsf{L}(a_n)| \geq n$.

For a numerical monoid other than $\mathbb{N}_0$ it is always the latter.

Distances

Let $L = \{\ell_1 < \ell_2 < \cdots < \ell_r\}$, then

$$\Delta(L) = \{\ell_2 - \ell_1, \ell_3 - \ell_2, \ldots, \ell_r - \ell_{r-1}\}.$$

For H BF-monoid, let

$$\Delta(H) = \bigcup_{a \in H} \Delta(\mathsf{L}(a))$$

the set distances of H.
And,

$$\min \Delta(H)$$

the minimal distance of H.

Elasticities

Consider for $k \in \mathbb{N}$

$$\rho_k(H) = \sup\{\sup L \colon k \in L, L \in \mathcal{L}(H)\}$$

And, $\rho(H) = \sup_k \rho_k(H)/k$.
Or, for $a \in H \setminus \{0\}$ let

- $\rho(a) = \sup \mathsf{L}(a) / \min \mathsf{L}(a)$,
- $R(H) = \{\rho(a) \colon a \in H \setminus \{0\}\}$ and
- $\rho(H) = \sup R(H)$.

Some results on elasticities

Let $H = \langle n_1 < \cdots < n_t \rangle$ a numerical monoid.

- Then $\rho(H) = n_t/n_1$, and this is the unique accumulation point of $R(H)$. (Chapman, Holden, Moore, 2006)
- More precisely, $R(H)$ is the union of a finite set and $n_1 n_t$ monotone increasing sequences converging to $\rho(H)$. (Barron, O'Neill, Pelayo, 2017)

Some results on distances

Let $H = \langle n_1 < \cdots < n_t \rangle$ a numerical monoid.

$$\min \Delta(H) = \gcd \Delta(H) = \gcd\{n_{i+1} - n_i \colon i = 1, \ldots, t-1\}$$

Let $H = \langle n_1 < n_2 \rangle$ then $\Delta(H) = \{n_2 - n_1\}$.

- Let $d \geq 1$ and $t \geq 2$, then there exists a numerical monoid H with $\Delta(H) = \{d, 2d, \ldots, td\}$. (Bowles, Chapman, Kaplan, Reiser 2007)
- Let $d \geq 1$ and $t \geq 2$, then there exists a numerical monoid H with $\Delta(H) = \{d, td\}$. (Colton, Kaplan 2017)

Which sets are sets of lengths

Let H be a BF monoid. We know

$$\mathcal{L}(H) \subset \{\{0\}, \{1\}\} \cup \mathbb{P}_{\text{fin}}(\mathbb{N}_{\geq 2}).$$

Is there a tighter ambient set?
For a particular monoid or also for a particular class of monoids?

- In general, no. For example, for monoids of zero-sum sequences over infinite abelian groups or for $\text{Int}(\mathbb{Z})$ one has $\mathcal{L}(H) = \{\{0\}, \{1\}\} \cup \mathbb{P}_{\text{fin}}(\mathbb{N}_{\geq 2})$.
- For numerical monoids?

Which sets are sets of lengths

Let H be a numerical monoid. Since $\rho(H) = n_t/n_1$, of course

$$\mathcal{L}(H) \subsetneq \{\{0\}, \{1\}\} \cup \mathbb{P}_{\text{fin}}(\mathbb{N}_{\geq 2}).$$

But the restriction depends on the specific H.
Question: Is there some $L \in \{\{0\}, \{1\}\} \cup \mathbb{P}_{\text{fin}}(\mathbb{N}_{\geq 2})$ that does not appear in $\mathcal{L}(H)$ for any numerical monoid at all?
Answer: No. ‘Every’ set is a set of length for some numerical monoid. (Geroldinger, S.). And even something more precise holds.

Some more notation

Let H be an additively written monoid. The (additively written) free abelian monoid $\mathsf{Z}(H) = \mathcal{F}(\mathcal{A}(H_{\text{red}}))$ is called the *factorization monoid* of H and the canonical epimorphism $\pi\colon \mathsf{Z}(H) \to H_{\text{red}}$ is the factorization homomorphism. For $a \in H$ and $k \in \mathbb{N}$,

- $\mathsf{Z}_H(a) = \mathsf{Z}(a) = \pi^{-1}(a + H^\times) \subset \mathsf{Z}(H)$ set of factorizations of a
- $\mathsf{Z}_{H,k}(a) = \mathsf{Z}_k(a) = \{z \in \mathsf{Z}(a) \mid |z| = k\}$ set of factorizations of a of length k
- $\mathsf{L}_H(a) = \mathsf{L}(a) = \big\{|z| \;\big|\; z \in \mathsf{Z}(a)\big\} \subset \mathbb{N}_0$ is the set of lengths of a.

Main result

Theorem (Geroldinger, S.)

Let $L \subset \mathbb{N}_{\geq 2}$ be a finite nonempty set and $f \colon L \to \mathbb{N}$ a map. Then there exist a numerical monoid H and a squarefree element $a \in H$ such that

$$\mathsf{L}(a) = L \quad \textit{and} \quad |\mathsf{Z}_k(a)| = f(k) \quad \textit{for every } k \in L\,.$$

A result for rings

Corollary

Let K be a field, $L \subset \mathbb{N}_{\geq 2}$ a finite nonempty set, and $f \colon L \to \mathbb{N}$ a map. Then there is a numerical monoid H and a squarefree element $g \in K[H]$ such that

$$\mathsf{L}_{K[H]}(g) = L \quad \textit{and} \quad |\mathsf{Z}_{K[H],k}(g)| = f(k) \quad \textit{for every } k \in L\,.$$

Idea of proof

- Construct the monoid recursively.
- Add new irreducibles to create new factorizations.
- Problem: there might be no space left.
- Solution: change the scale.

To do this cleanly we work with Puiseux monoids, that is, submonoids of $(\mathbb{Q}_{\geq 0}, +)$.

A dual question

We just saw

$$\bigcup_{H \text{ numerical mon.}} \mathcal{L}(H) = \{\{0\}, \{1\}\} \cup \mathbb{P}_{\text{fin}}(\mathbb{N}_{\geq 2}).$$

What about

$$\bigcap_{H \text{ numerical mon.}} \mathcal{L}(H) \quad ?$$

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