ON STRONGLY PRIMARY MONOIDS AND DOMAINS

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Consider a local one-dimensional Mori (in particular, noetherian) domain (R, M). Since R is Mori, we see that each non-zero principal ideal of R contains a power of the maximal ideal M. Domains satisfying this property are called *strongly primary* domains.

A monoid in this talk is a cancellative semigroup with 1 which is not a group. The monoid H is *primary* if for any two non-units $a, b \in H$, the s-ideal aH contains a power of bH. The monoid H is *strongly primary* if each non-unit of H contains a power of $\mathfrak{m}$ (the set of non-units of H).

It is well-known and easy to prove that if H is a strongly primary monoid, then $\bigcap_{n\geq 1} \mathfrak{m}^n = \emptyset$. In particular, H is archimedean, that is $\bigcap_{n\geq 1} s^n H = \emptyset$ for each non-unit $s \in H$. A strongly primary monoid is atomic.

If H is a strongly primary monoid, and $s \in H$, we denote by $\mathcal{M}(s)$ the smallest positive integer k such that $M^k \subseteq sH$. The minimal length of a factorization of an element $s \in \mathfrak{m}$ is denoted by $\Lambda(s)$. We let

$$\Lambda(H) = \sup\{\Lambda(s) \mid s \in \mathfrak{m}\}.$$

We let

$$\widehat{H} = \{x \in \mathsf{q}(H) \mid (\exists d \in H)(\forall n \in \mathbb{N})(dx^n \in H),$$

the complete integral closure of H.

For a domain R we define $\widehat{R}$ in a similar way, requiring that $d \neq 0$, thus $\widehat{R} = \widehat{R^\bullet} \cup \{0\}$, where $R^\bullet$ is the multiplicative monoid $R \setminus \{0\}$.

The purpose of this talk is to prove Theorem 1. Four auxiliary results needed in the proof are collected in Lemma 2. Theorem 1 provides a positive answer to Geroldinger's question whether a local one dimensional Mori domain is locally tame. Indeed, if H is an atomic monoid such that $\Lambda(H) < \infty$, then H is locally tame. On the other hand, it was already known that a local one dimensional Mori domain with nonzero conductor $(R\colon \widehat{R})$ is locally tame.

Theorem 1. *Let R be a strongly primary domain. If $(R\colon \widehat{R}) = \{0\}$, then $\Lambda(R) < \infty$.*

Proof. Assume that $\Lambda(R) = \infty$. Let $\mathfrak{n} = \widehat{H} \setminus (\widehat{H})^{\times}$. Choose a nonzero element $c \in M$. Let n be a positive integer. Since $(R\colon \widehat{R}) = \{0\}$, we infer that $(c^n\mathfrak{n})\widehat{R} \not\subseteq R$, whence there exists an element $x \in \mathfrak{n}$ such that $c^n x \notin R$.

By Lemma 2 (4), we have $x^i \in M$ for all sufficiently large $i \in \mathbb{N}$ (the set of nonnegative integers). Let $i \in \mathbb{N}$ be maximal such that $c^n x^i \notin M$, thus $i > 0$, and $c^n x^j \in M$ for $j > i$. Since the monoid $R^\bullet$ is primary, there exists a minimal positive integer k such that $c^k x^i \in M$, thus $k > n$, and $c^k x^j \in M$, for all $j \geq i$. Set $y = c^{k-n}x^i$. Thus $c^n y^j \in M$ for all $j \geq 1$, $\quad c^{n-1}y \notin M$, and $c^{n-1}y^j \in M$ for all $j > 1$. There exists an integer $e \in \mathbb{N}$ such that $y^e \in M$. Hence

$$(1-y)(1+y+\cdots+y^{e-1}) = 1-y^e \in R^{\times},$$

Thus

$$c^n(1-y) \in R, \qquad \text{and} \quad \frac{c^n}{1-y} = c^n\frac{1+y+\cdots+y^{e-1}}{1-y^e} \in R.$$

We see that $c^n(1-y)$ and $\frac{c^n}{1-y}$ are not divisible by c in R. We have

$$c^{2n} = \Big(c^n(1-y)\Big)\left(\frac{c^n}{1-y}\right).$$

Thus c^{2n} is a product of two elements that are not divisible by c in R, whence $\Lambda(c^{2n}) < \mathcal{M}(c) + \mathcal{M}(c)$. By Lemma 2 (2) we conclude that $\Lambda(R) < \infty$. $\square$

The next lemma is formulated for monoids, but it is clear that it can also be applied to domains.

Lemma 2. *Let $(H, \mathfrak{m})$ be a strongly primary monoid.*

1. *If $x \in \mathsf{q}(H)$ with $x^{-1} \notin H$, then $\Lambda(H \setminus xH) < \mathcal{M}(x)$.*
2. *We have $\Lambda(H) < \infty$ if and only if there is an element $c \in \mathfrak{m}$ with $\Lambda(\{c^m \mid m \in \mathbb{N}\}) < \infty$.*
3. *If there is an element $x \in \mathfrak{n}$ (the set of non-units in $\widehat{H}$), such that no power of x belongs to $\mathfrak{n}$, then $\Lambda(H) < \infty$.*
4. *If $\Lambda(H) = \infty$, then for every $x \in \mathfrak{n}$, we have $x^n \in \mathfrak{m}$ for all sufficiently large $n \in \mathbb{N}$.*

Proof.

(1) If $a \in H \setminus xH$, then $a \notin \mathfrak{m}^{\mathcal{M}(x)}$ as $\mathfrak{m}^{\mathcal{M}(x)} \subseteq xH$. Thus $\Lambda(H \setminus xH) < \mathcal{M}(x)$

(2) Let $c \in \mathfrak{m}$ such that $\Lambda(\{c^m \mid m \in \mathbb{N}\}) < \infty$. Let $a \in H$. . Since H is archimedean, there is an $n \in \mathbb{N}_0$ such that $a = c^n b$, where $b \in H$ is not divisible by c. Now (1) implies that

$$\Lambda(a) \leq \Lambda(c^n) + \Lambda(b) < \Lambda(\{c^m \mid m \in \mathbb{N}\}) + \mathcal{M}(c)\,.$$

Thus $\Lambda(H) < \infty$. The reverse implication is trivial.

(3) Let $d \in \mathfrak{m}$ such that $dx^n \in H$ for all $n \in \mathbb{N}$. Let $c \in \mathfrak{m}$. If $c \notin dH$, then $\Lambda(c) < \mathcal{M}(d)$ by (1). Suppose that $c \in dH$. Since x is not invertible in $\widehat{H}$, there is an integer $n \in \mathbb{N}$ such that $(cd^{-1})x^{-n} \in H$ and $(cd^{-1})x^{-(n+1)} \notin H$. Thus $(cd^{-1})x^{-n} \notin xH$. so $\Lambda((cd^{-1})x^{-n}) < \mathcal{M}(x)$ by 1. Since $dx^n \notin dH$ we obtain by (1) that

$$\Lambda(c) = \Lambda((dx^n)((cd^{-1})x^{-n})) < \mathcal{M}(d) + \mathcal{M}(x).$$

Hence $\Lambda(H) < \mathcal{M}(d) + \mathcal{M}(x) < \infty$.

(4) Let $x \in \mathfrak{n}$. Since $\Lambda(H) = \infty$, by item (3) there is a $k \in \mathbb{N}$ such that $x^k \in \mathfrak{m}$. Since H is primary, there is a $q_0 \in \mathbb{N}$ such that $x^{q_0 k + r} = (x^k)^{q_0} x^r \in \mathfrak{m}$ for all $r \in [0, k-1]$. If $n \in \mathbb{N}$ with $n \geq q_0 k$, then $n = qk + r$, where $q \geq q_0$ and $r \in [0, k-1]$, and $x^n = x^{k(q-q_0)} x^{q_0 k + r} \in \mathfrak{m}$.

□

Theorem 1 uses the additive structure of the domain R, although it is formulated in multiplcative terms. This is not incidental. Indeed, this theorem is false for strongly primary monoids, as shown in Proposition 3.7 of the article

A. Geroldinger, W. Hassler, and G. Lettl, *On the arithmetic of strongly primary monoids*, Semigroup Forum **75** (2007), 567 – 587.

Next, we formulate without proof three more theorems. Some parts of them are well-known.

Theorem 3.

(a) *Let $(H, \mathfrak{m})$ be a strongly primary monoid. Then each of the first 7 conditions below implies its successor. Moreover, the first two condtions are equivalent:*

(1) *H is globally tame.*

(2) $\bigcap_{a \in \mathcal{A}(H)} aH \neq \emptyset$.

(3) *There is a $k \in \mathbb{N}$ such that $\mathfrak{m}^k \subseteq aH$ for every $a \in \mathcal{A}(H)$.*

(4) $\rho(H) < \infty$.

(5) *$\rho_k(H) < \infty$ for all $k \in \mathbb{N}$.*

(6) $\Lambda(H)=\infty$.
(7) $\widehat{H}$ *is a primary valuation monoid.*
(8) $\widehat{H}$ *is a valuation monoid.*

If $\mathsf{f}=(H:\widehat{H})\neq\emptyset$*, then all conditions are equivalent, including conditions (9)-(10) below:*

(9) $\mathfrak{fm}\subseteq\bigcap_{a\in\mathcal{A}(H)}a\widehat{H}$.
(10) $\mathfrak{fm}^2\subseteq\bigcap_{a\in\mathcal{A}(H)}aH$.

(b) *Let Rbe a strongly primary domain. Then all conditions of item (a) are equivalent if we replace H by R and add the requirement* $f=(R\colon\widehat{R})\neq\emptyset$ *to conditions (7) and (8):*

(7[]) $\widehat{R}$ *is a primary valuation domain and* $f=(R\colon\widehat{R})\neq\emptyset$.
(8) $\widehat{R}$ *is a valuation domain and* $f=(R\colon\widehat{R})\neq\emptyset$.

Theorem 4.

(1) *Let H be a strongly primary monoid. Then H is locally tame if it satisfies one of the following two conditions:*
- $\Lambda(H)<\infty$.
- $\Lambda(H)=\infty$ *and* $(\widehat{H}\colon H)\neq(0)$

Moreover, if $(\widehat{H}:H)\neq(0)$*, then all conditions (1)-(10) of Theorem 3 are equivalent.*

(2) *A strongly primary domain is locally tame, and conditions (1)-(7), (8'), (9'), (10) of Theorem 3 are equivalent.*

Theorem 5.

(1) *Let H be a strongly primary monoid.*
  (a) *H is globally tame if* $\Lambda(H)=\infty$ *and* $(H\colon\widehat{H})\neq\emptyset$.
  (b) *H is not globally tame if* $\Lambda(H)<\infty$

(2) *A strongly primary domain R is globally tame if and only if* $\Lambda(R)=\infty$*. A globally tame domain satisfies all the conditions (1)-(10) of Theorem 3.*