Progress with the Prime Ideal Principle

Manuel L. Reyes
Bowdoin College

Conference on Rings and Factorizations — University of Graz
February 22, 2018

Today's themes

Major questions that I will address in this talk:

(1) Is there an underlying framework behind the many "maximal implies prime" results for ideals in commutative rings?

(2) Are there similar "maximal implies prime" results for right ideals in noncommutative rings?

(3) Is there an underlying framework for the (fewer) "maximal implies prime" results for two-sided ideals in noncommutative rings?

1. When “maximal implies prime” in commutative algebra
2. When “maximal implies prime” for one-sided ideals
3. A two-sided Prime Ideal Principle

Motivating results in commutative algebra

Cohen's Theorem (1950): A commutative ring is noetherian iff all of its prime ideals are finitely generated.

Kaplansky's Theorem (1949): For a commutative ring R, TFAE:

- R is a principal ideal ring (PIR);
- R is noetherian and every maximal ideal of R is principal;
- every prime ideal of R is principal. ($\leftarrow$ Used Cohen's Theorem.)

Motivating results in commutative algebra

Cohen's Theorem (1950): A commutative ring is noetherian iff all of its prime ideals are finitely generated.

Kaplansky's Theorem (1949): For a commutative ring R, TFAE:

- R is a principal ideal ring (PIR);
- R is noetherian and every maximal ideal of R is principal;
- every prime ideal of R is principal. ($\leftarrow$ Used Cohen's Theorem.)

Typical proof of the "hard part":

- Suppose R has an ideal P that is not f.g. (resp. principal).
- Using Zorn's Lemma, pass to a "maximal counterexample": an ideal $P \supseteq I$ maximal w.r.t. not being f.g. or principal.
- Prove that such maximal P is prime.

The "maximal implies prime" phenomenon

There is an array of related results within commutative algebra:

Theorems

In a commutative ring R, an ideal I maximal with respect to

- *being proper ($\neq R$)*

is prime.

The "maximal implies prime" phenomenon

There is an array of related results within commutative algebra:

Theorems

In a commutative ring R, an ideal I maximal with respect to

- *being disjoint from a fixed multiplicative set $S \subseteq R$*

is prime.

The "maximal implies prime" phenomenon

There is an array of related results within commutative algebra:

Theorems

In a commutative ring R, an ideal I maximal with respect to

- *being non-finitely generated*

is prime.

The "maximal implies prime" phenomenon

There is an array of related results within commutative algebra:

Theorems

In a commutative ring R, an ideal I maximal with respect to

- *being non-principal*

is prime.

The "maximal implies prime" phenomenon

There is an array of related results within commutative algebra:

Theorems

In a commutative ring R, an ideal I maximal with respect to

- *being proper ($\neq R$)*
- *being disjoint from a fixed multiplicative set $S \subseteq R$*
- *being non-finitely generated*
- *being non-principal*

is prime.

A natural question: (Joint work with T. Y. Lam) What is common to all of these properties?

Idea: If a family $\mathcal{F}$ of ideals has a sutiable closure property, then P maximal w.r.t. $P \notin \mathcal{F}$ will be prime.

Oka families of ideals in commutative rings

Recall that $(I : a) = \{r \in R : ar \in I\} \trianglelefteq R$.

Def: A family $\mathcal{F}$ of ideals in a commutative ring R is an Oka family if:

1. The ideal $R \in \mathcal{F}$, and
2. For all $I \trianglelefteq R$ and $a \in R$,

$$(I, a),\ (I : a) \in \mathcal{F} \implies I \in \mathcal{F}.$$

Oka families of ideals in commutative rings

Recall that $(I : a) = \{r \in R : ar \in I\} \trianglelefteq R$.

Def: A family $\mathcal{F}$ of ideals in a commutative ring R is an Oka family if:

1. The ideal $R \in \mathcal{F}$, and
2. For all $I \trianglelefteq R$ and $a \in R$,

$$(I, a),\ (I : a) \in \mathcal{F} \implies I \in \mathcal{F}.$$

Why "Oka" families? The complex analyst K. Oka proved a lemma (1951), generalized by M. Nagata (1956) to arbitrary commutative rings:

Proposition ("Oka's Lemma") If an ideal I and an element a of some commutative ring R are such that (I, a) and $(I : a)$ are finitely generated, then I itself is finitely generated.

A Prime Ideal Principle in commutative algebra

Notation: For a family $\mathcal{F}$ of right ideals in a ring R,

- $\mathcal{F}' := \{I_R \subseteq R : I \notin \mathcal{F}\}$, the complement of $\mathcal{F}$;
- $\mathrm{Max}(\mathcal{F}')$ denotes the set of right ideals maximal in $\mathcal{F}'$.

The reason for Oka families:

Prime Ideal Principle [Lam, R. '08]: Let $\mathcal{F}$ be an Oka family of ideals in a commutative ring R. Then any ideal $I \in \mathrm{Max}(\mathcal{F}')$ is prime.

A Prime Ideal Principle in commutative algebra

Notation: For a family $\mathcal{F}$ of right ideals in a ring R,

- $\mathcal{F}' := \{I_R \subseteq R : I \notin \mathcal{F}\}$, the complement of $\mathcal{F}$;
- $\mathrm{Max}(\mathcal{F}')$ denotes the set of right ideals maximal in $\mathcal{F}'$.

The reason for Oka families:

Prime Ideal Principle [Lam, R. '08]: Let $\mathcal{F}$ be an Oka family of ideals in a commutative ring R. Then any ideal $I \in \mathrm{Max}(\mathcal{F}')$ is prime.

Proof: Notice that $I \subsetneq R$ because $R \in \mathcal{F}$.

A Prime Ideal Principle in commutative algebra

Notation: For a family $\mathcal{F}$ of right ideals in a ring R,

- $\mathcal{F}' := \{I_R \subseteq R : I \notin \mathcal{F}\}$, the complement of $\mathcal{F}$;
- $\mathrm{Max}(\mathcal{F}')$ denotes the set of right ideals maximal in $\mathcal{F}'$.

The reason for Oka families:

Prime Ideal Principle [Lam, R. '08]: Let $\mathcal{F}$ be an Oka family of ideals in a commutative ring R. Then any ideal $I \in \mathrm{Max}(\mathcal{F}')$ is prime.

Proof: Notice that $I \subsetneq R$ because $R \in \mathcal{F}$.
Assume toward a contradiction that there exist $a, b \in R$ with $ab \in I$ but $a, b \notin I$.

A Prime Ideal Principle in commutative algebra

Notation: For a family $\mathcal{F}$ of right ideals in a ring R,

- $\mathcal{F}' := \{I_R \subseteq R : I \notin \mathcal{F}\}$, the complement of $\mathcal{F}$;
- $\mathrm{Max}(\mathcal{F}')$ denotes the set of right ideals maximal in $\mathcal{F}'$.

The reason for Oka families:

Prime Ideal Principle [Lam, R. '08]: Let $\mathcal{F}$ be an Oka family of ideals in a commutative ring R. Then any ideal $I \in \mathrm{Max}(\mathcal{F}')$ is prime.

Proof: Notice that $I \subsetneq R$ because $R \in \mathcal{F}$.
Assume toward a contradiction that there exist $a, b \in R$ with $ab \in I$ but $a, b \notin I$. Because $a \notin I$, $(I, a) \supsetneq I$.

A Prime Ideal Principle in commutative algebra

Notation: For a family $\mathcal{F}$ of right ideals in a ring R,

- $\mathcal{F}' := \{I_R \subseteq R : I \notin \mathcal{F}\}$, the complement of $\mathcal{F}$;
- $\mathrm{Max}(\mathcal{F}')$ denotes the set of right ideals maximal in $\mathcal{F}'$.

The reason for Oka families:

Prime Ideal Principle [Lam, R. '08]: Let $\mathcal{F}$ be an Oka family of ideals in a commutative ring R. Then any ideal $I \in \mathrm{Max}(\mathcal{F}')$ is prime.

Proof: Notice that $I \subsetneq R$ because $R \in \mathcal{F}$.
Assume toward a contradiction that there exist $a, b \in R$ with $ab \in I$ but $a, b \notin I$. Because $a \notin I$, $(I, a) \supsetneq I$.
Also $I \subseteq (I : a)$,

A Prime Ideal Principle in commutative algebra

Notation: For a family $\mathcal{F}$ of right ideals in a ring R,

- $\mathcal{F}' := \{I_R \subseteq R : I \notin \mathcal{F}\}$, the complement of $\mathcal{F}$;
- $\mathrm{Max}(\mathcal{F}')$ denotes the set of right ideals maximal in $\mathcal{F}'$.

The reason for Oka families:

Prime Ideal Principle [Lam, R. '08]: Let $\mathcal{F}$ be an Oka family of ideals in a commutative ring R. Then any ideal $I \in \mathrm{Max}(\mathcal{F}')$ is prime.

Proof: Notice that $I \subsetneq R$ because $R \in \mathcal{F}$.
Assume toward a contradiction that there exist $a, b \in R$ with $ab \in I$ but $a, b \notin I$. Because $a \notin I$, $(I, a) \supsetneq I$.
Also $I \subseteq (I : a)$, and $(I : a) \supsetneq I$ because $b \in (I : a) \setminus I$.

A Prime Ideal Principle in commutative algebra

Notation: For a family $\mathcal{F}$ of right ideals in a ring R,

- $\mathcal{F}' := \{I_R \subseteq R : I \notin \mathcal{F}\}$, the complement of $\mathcal{F}$;
- $\mathrm{Max}(\mathcal{F}')$ denotes the set of right ideals maximal in $\mathcal{F}'$.

The reason for Oka families:

Prime Ideal Principle [Lam, R. '08]: Let $\mathcal{F}$ be an Oka family of ideals in a commutative ring R. Then any ideal $I \in \mathrm{Max}(\mathcal{F}')$ is prime.

Proof: Notice that $I \subsetneq R$ because $R \in \mathcal{F}$.
Assume toward a contradiction that there exist $a, b \in R$ with $ab \in I$ but $a, b \notin I$. Because $a \notin I$, $(I, a) \supsetneq I$.
Also $I \subseteq (I : a)$, and $(I : a) \supsetneq I$ because $b \in (I : a) \setminus I$.
By maximality of I, we have $(I, a),\ (I : a) \in \mathcal{F}$.

A Prime Ideal Principle in commutative algebra

Notation: For a family $\mathcal{F}$ of right ideals in a ring R,

- $\mathcal{F}' := \{I_R \subseteq R : I \notin \mathcal{F}\}$, the complement of $\mathcal{F}$;
- $\text{Max}(\mathcal{F}')$ denotes the set of right ideals maximal in $\mathcal{F}'$.

The reason for Oka families:

Prime Ideal Principle [Lam, R. '08]: Let $\mathcal{F}$ be an Oka family of ideals in a commutative ring R. Then any ideal $I \in \text{Max}(\mathcal{F}')$ is prime.

Proof: Notice that $I \subsetneq R$ because $R \in \mathcal{F}$.
Assume toward a contradiction that there exist $a, b \in R$ with $ab \in I$ but $a, b \notin I$. Because $a \notin I$, $(I, a) \supsetneq I$.
Also $I \subseteq (I : a)$, and $(I : a) \supsetneq I$ because $b \in (I : a) \setminus I$.
By maximality of I, we have (I, a), $(I : a) \in \mathcal{F}$.
Since $\mathcal{F}$ is an Oka family, $I \in \mathcal{F}$,

A Prime Ideal Principle in commutative algebra

Notation: For a family $\mathcal{F}$ of right ideals in a ring R,

- $\mathcal{F}' := \{I_R \subseteq R : I \notin \mathcal{F}\}$, the complement of $\mathcal{F}$;
- $\mathrm{Max}(\mathcal{F}')$ denotes the set of right ideals maximal in $\mathcal{F}'$.

The reason for Oka families:

Prime Ideal Principle [Lam, R. '08]: Let $\mathcal{F}$ be an Oka family of ideals in a commutative ring R. Then any ideal $I \in \mathrm{Max}(\mathcal{F}')$ is prime.

Proof: Notice that $I \subsetneq R$ because $R \in \mathcal{F}$.
Assume toward a contradiction that there exist $a, b \in R$ with $ab \in I$ but $a, b \notin I$. Because $a \notin I$, $(I, a) \supsetneq I$.
Also $I \subseteq (I : a)$, and $(I : a) \supsetneq I$ because $b \in (I : a) \setminus I$.
By maximality of I, we have (I, a), $(I : a) \in \mathcal{F}$.
Since $\mathcal{F}$ is an Oka family, $I \in \mathcal{F}$, a contradiction.

Proof of Oka's lemma

A sample proof that a family is Oka:

Oka's Lemma (reformulated): Let R be a commutative ring. The family $\mathcal{F}$ of finitely generated ideals of R is Oka.

Proof: Clearly $R \in \mathcal{F}$. Suppose $I \trianglelefteq R$, $a \in R$ with $(I, a), (I : a) \in \mathcal{F}$.

Proof of Oka's lemma

A sample proof that a family is Oka:

Oka's Lemma (reformulated): Let R be a commutative ring. The family $\mathcal{F}$ of finitely generated ideals of R is Oka.

Proof: Clearly $R \in \mathcal{F}$. Suppose $I \trianglelefteq R$, $a \in R$ with $(I, a), (I : a) \in \mathcal{F}$. Then

$$(I, a) = (x_1 + ar_1, \ldots, y_n + ar_n) \qquad \text{for some} \qquad y_i \in I, r_i \in R.$$

Proof of Oka's lemma

A sample proof that a family is Oka:

Oka's Lemma (reformulated): Let R be a commutative ring. The family $\mathcal{F}$ of finitely generated ideals of R is Oka.

Proof: Clearly $R \in \mathcal{F}$. Suppose $I \trianglelefteq R$, $a \in R$ with $(I, a), (I : a) \in \mathcal{F}$. Then

$$(I, a) = (x_1 + ar_1, \ldots, y_n + ar_n) \qquad \text{for some} \qquad y_i \in I, r_i \in R.$$

Fix $x \in I$; then we may write

$$x \in I \subseteq (I, a) \implies x = \sum (y_i + ar_i)s_i = y + ar$$

for some $y \in (y_1, \ldots, y_n) \subseteq I$ and $r \in R$.

Proof of Oka's lemma

A sample proof that a family is Oka:

Oka's Lemma (reformulated): Let R be a commutative ring. The family $\mathcal{F}$ of finitely generated ideals of R is Oka.

Proof: Clearly $R \in \mathcal{F}$. Suppose $I \trianglelefteq R$, $a \in R$ with $(I, a), (I : a) \in \mathcal{F}$. Then

$$(I, a) = (x_1 + ar_1, \dots, y_n + ar_n) \qquad \text{for some} \qquad y_i \in I, r_i \in R.$$

Fix $x \in I$; then we may write

$$x \in I \subseteq (I, a) \implies x = \sum (y_i + ar_i) s_i = y + ar$$

for some $y \in (y_1, \dots, y_n) \subseteq I$ and $r \in R$. But then

$$ar = y - x \in I \implies r \in (I : a).$$

Proof of Oka's lemma

A sample proof that a family is Oka:

Oka's Lemma (reformulated): Let R be a commutative ring. The family $\mathcal{F}$ of finitely generated ideals of R is Oka.

Proof: Clearly $R \in \mathcal{F}$. Suppose $I \trianglelefteq R$, $a \in R$ with $(I, a), (I : a) \in \mathcal{F}$. Then

$$(I, a) = (x_1 + ar_1, \ldots, y_n + ar_n) \qquad \text{for some} \qquad y_i \in I, r_i \in R.$$

Fix $x \in I$; then we may write

$$x \in I \subseteq (I, a) \implies x = \sum (y_i + ar_i)s_i = y + ar$$

for some $y \in (y_1, \ldots, y_n) \subseteq I$ and $r \in R$. But then

$$ar = y - x \in I \implies r \in (I : a).$$

It follows that

$$I \subseteq (y_1, \ldots, y_n) + a(I : a) \subseteq I.$$

Proof of Oka's lemma

A sample proof that a family is Oka:

Oka's Lemma (reformulated): Let R be a commutative ring. The family $\mathcal{F}$ of finitely generated ideals of R is Oka.

Proof: Clearly $R \in \mathcal{F}$. Suppose $I \trianglelefteq R$, $a \in R$ with $(I, a), (I : a) \in \mathcal{F}$. Then

$$(I, a) = (x_1 + ar_1, \ldots, y_n + ar_n) \qquad \text{for some} \qquad y_i \in I, r_i \in R.$$

Fix $x \in I$; then we may write

$$x \in I \subseteq (I, a) \implies x = \sum (y_i + ar_i)s_i = y + ar$$

for some $y \in (y_1, \ldots, y_n) \subseteq I$ and $r \in R$. But then

$$ar = y - x \in I \implies r \in (I : a).$$

It follows that

$$I \subseteq (y_1, \ldots, y_n) + a(I : a) \subseteq I.$$

Thus $I = I + a(I : a)$, is finitely generated because I and $(I : a)$ are. QED

Examples of Oka families

Examples

For a commutative ring R, the following families $\mathcal{F}$ of ideals are Oka families:

- $\{R\}$;
- The ideals intersecting a fixed multiplicative set $S \subseteq R$;
- The finitely generated ideals of R (Oka's Lemma);
- The principal ideals of R.

Examples of Oka families

Examples

For a commutative ring R, the following families $\mathcal{F}$ of ideals are Oka families:

- $\{R\}$;
- The ideals intersecting a fixed multiplicative set $S \subseteq R$;
- The finitely generated ideals of R (Oka's Lemma);
- The principal ideals of R.

Because these are all Oka families, the Prime Ideal Principle (PIP) unifies the "maximal implies prime" results mentioned above!

Further examples of Oka families

In a general commutative ring R, the following are Oka families:

1. Any family of invertible ideals that is closed under products;
2. $\mathcal{F} = \{(s) \mid s \in S\}$ for any multiplicative set $S \subseteq R$;
3. the set of multiplication ideals (I such that $J \subseteq I \implies J = I(J : I)$)
4. the set of ideals I such that $I \supseteq I^2 \supseteq I^3 \supseteq \cdots$ stabilizes;
5. the set of idempotent ideals;
6. the ideals that are direct summands of R.

Further examples of Oka families

In a general commutative ring R, the following are Oka families:

1. Any family of invertible ideals that is closed under products;
2. $\mathcal{F} = \{(s) \mid s \in S\}$ for any multiplicative set $S \subseteq R$;
3. the set of multiplication ideals (I such that $J \subseteq I \implies J = I(J : I)$)
4. the set of ideals I such that $I \supseteq I^2 \supseteq I^3 \supseteq \cdots$ stabilizes;
5. the set of idempotent ideals;
6. the ideals that are direct summands of R.

Of course, an ideal maximal in the complement of any of these families is prime, but:

Theorem: An ideal in $\mathrm{Max}(\mathcal{F}')$ for any of the families $\mathcal{F}$ of (4)–(6) above must be a maximal ideal.

Oka families from the module perspective

What does the Oka property really mean? For me the best answer is module-theoretic.

Ideals correspond (essentially) bijectively to cyclic modules via $I \mapsto M = R/I$ and $M \mapsto I = \mathrm{ann}(M)$. Extend to families:

$$\mathcal{F} \mapsto \mathcal{C}_{\mathcal{F}} = \{{}_R M \mid M \cong R/I \text{ for some } I \in \mathcal{F}\}.$$

Oka families from the module perspective

What does the Oka property really mean? For me the best answer is module-theoretic.

Ideals correspond (essentially) bijectively to cyclic modules via $I \mapsto M = R/I$ and $M \mapsto I = \mathrm{ann}(M)$. Extend to families:

$$\mathcal{F} \mapsto \mathcal{C}_{\mathcal{F}} = \{{}_R M \mid M \cong R/I \text{ for some } I \in \mathcal{F}\}.$$

Theorem [LR '08]: A family $\mathcal{F}$ of ideals is Oka if and only if the class $\mathcal{C}_{\mathcal{F}}$ of cyclic modules is closed under cyclic extensions. That is, $\mathcal{F}$ is Oka if and only if, for every short exact sequence of cyclic modules

$$0 \to R/J \to R/I \to R/K \to 0,$$

if $J, K \in \mathcal{F}$ then $I \in \mathcal{F}$.

Oka families from the module perspective

What does the Oka property really mean? For me the best answer is module-theoretic.

Ideals correspond (essentially) bijectively to cyclic modules via $I \mapsto M = R/I$ and $M \mapsto I = \mathrm{ann}(M)$. Extend to families:

$$\mathcal{F} \mapsto \mathcal{C}_{\mathcal{F}} = \{{}_R M \mid M \cong R/I \text{ for some } I \in \mathcal{F}\}.$$

Theorem [LR '08]: A family $\mathcal{F}$ of ideals is Oka if and only if the class $\mathcal{C}_{\mathcal{F}}$ of cyclic modules is closed under cyclic extensions. That is, $\mathcal{F}$ is Oka if and only if, for every short exact sequence of cyclic modules

$$0 \to R/J \to R/I \to R/K \to 0,$$

if $J, K \in \mathcal{F}$ then $I \in \mathcal{F}$.

Idea: Any short exact sequence as above is isomorphic to one of the form

$$0 \to R/(I : a) \to R/I \to R/(I, a) \to 0$$

Examples of cyclic module classes

Examples:

- $\mathcal{F} = \{I \mid I \cap S \neq \varnothing\} \rightsquigarrow \mathcal{C}_{\mathcal{F}} = \{S\text{-torsion cyclic modules}\}$
- $\mathcal{F} = \{\text{f.g. ideals}\} \rightsquigarrow \mathcal{C}_{\mathcal{F}} = \{\text{finitely presented cyclic modules}\}$
- $\mathcal{F} = \{\text{principal ideals}\} \rightsquigarrow \mathcal{C}_{\mathcal{F}} = \{\text{cyclically presented modules}\}$

Examples of cyclic module classes

Examples:

- $\mathcal{F} = \{I \mid I \cap S \neq \varnothing\} \rightsquigarrow \mathcal{C}_{\mathcal{F}} = \{S\text{-torsion cyclic modules}\}$
- $\mathcal{F} = \{\text{f.g. ideals}\} \rightsquigarrow \mathcal{C}_{\mathcal{F}} = \{\text{finitely presented cyclic modules}\}$
- $\mathcal{F} = \{\text{principal ideals}\} \rightsquigarrow \mathcal{C}_{\mathcal{F}} = \{\text{cyclically presented modules}\}$

This also gives us a way to recognize new Oka families, beginning with a class $\mathcal{C}$ of cyclic modules closed under (cyclic) extensions, and producing $\mathcal{F}_{\mathcal{C}} = \{I \mid R/I \in \mathcal{C}\}$.

Examples:

- $\mathcal{C} = \{\text{cyclic modules with gl.dim } \leq 1\} \rightsquigarrow \mathcal{F}_{\mathcal{C}} = \{\text{projective ideals}\}$.
- $\mathcal{C} = \{\text{flat cyclic modules}\} \rightsquigarrow \mathcal{F}_{\mathcal{C}} = \{\text{pure ideals}\}$
- $\mathcal{C} = \{\text{finite cyclic modules}\} \rightsquigarrow \mathcal{F}_{\mathcal{C}} = \{\text{ideals of finite index}\}$

Other closure properties of ideal families

Several other properties guarantee $\mathrm{Max}(\mathcal{F}') \subseteq \mathrm{Spec}(R)$.

We always assume that $R \in \mathcal{F}$. We say $\mathcal{F}$ is monoidal if it is closed under ideal products, and define the following properties:

- (P_1): $\mathcal{F}$ is monoidal and $J \in \mathcal{F}, J \subseteq I \implies I \in \mathcal{F}$
- (P_2): $\mathcal{F}$ is monoidal and $J \in \mathcal{F}, I^2 \subseteq J \subseteq I \implies I \in \mathcal{F}$
- (P_3): $\mathcal{F}$ closed under pairwise intersection and $J \in \mathcal{F}, J^2 \subseteq I \subseteq J \implies I \in \mathcal{F}$
- Ako: $(I, a), (I, b) \in \mathcal{F} \implies (I, ab) \in \mathcal{F}$

Other closure properties of ideal families

Several other properties guarantee $\mathrm{Max}(\mathcal{F}') \subseteq \mathrm{Spec}(R)$.

We always assume that $R \in \mathcal{F}$. We say $\mathcal{F}$ is monoidal if it is closed under ideal products, and define the following properties:

- (P_1): $\mathcal{F}$ is monoidal and $J \in \mathcal{F}, J \subseteq I \implies I \in \mathcal{F}$
- (P_2): $\mathcal{F}$ is monoidal and $J \in \mathcal{F}, I^2 \subseteq J \subseteq I \implies I \in \mathcal{F}$
- (P_3): $\mathcal{F}$ closed under pairwise intersection and $J \in \mathcal{F}, J^2 \subseteq I \subseteq J \implies I \in \mathcal{F}$
- Ako: $(I, a), (I, b) \in \mathcal{F} \implies (I, ab) \in \mathcal{F}$

We have the following logical dependence between the properties:

$$\begin{array}{ccccccc} (P_1) & \implies & (P_2) & \implies & (P_3) & \implies & \text{Oka} \\ & & & & \Downarrow & & \Downarrow \\ & & & & \text{Ako} & \implies & \text{PIP} \end{array}$$

Ideal families in various classes of rings

It can be an interesting problem to determine the structure of ideal families in certain classes of commutative rings. For instance:

Theorem [Lam, R. '09]: For commutative rings of these types, we have the following relations between ideal properties:

- von Neumann regular rings: Ako $\implies$ Oka
- Dedekind domains: Ako $\Leftrightarrow$ (P_3) $\implies$ Oka $\Leftrightarrow$ monoidal
- integral domain: Oka $\implies$ $\frac{1}{2}$-monoidal: $(a), B \in \mathcal{F} \implies aB \in \mathcal{F}$
- valuation domain: Oka $\iff$ $\frac{1}{2}$-monoidal

Ideal families in various classes of rings

It can be an interesting problem to determine the structure of ideal families in certain classes of commutative rings. For instance:

Theorem [Lam, R. '09]: For commutative rings of these types, we have the following relations between ideal properties:

- von Neumann regular rings: Ako $\implies$ Oka
- Dedekind domains: Ako $\Leftrightarrow (P_3) \implies$ Oka $\Leftrightarrow$ monoidal
- integral domain: Oka $\implies$ $\frac{1}{2}$-monoidal: $(a), B \in \mathcal{F} \implies aB \in \mathcal{F}$
- valuation domain: Oka $\iff$ $\frac{1}{2}$-monoidal

Might there be similar results in other nice classes of rings? For instance:

Question: How to characterize Oka/Ako families in Prüfer domains?

Separating the Ako and Oka properties

Ex: In the Dedekind domain $\mathbb{Z}$, the monoidal family

$$\mathcal{F} = \{(4^n) \mid n \geq 0\}$$

is Oka, but not Ako since it violates (P_3): $(4)^2 \subseteq (8) \subseteq (4)$ but $(8) \notin \mathcal{F}$.

While it seems "clear" that Ako $\not\Rightarrow$ Oka, all "natural" Ako families seem to be Oka, and several classes of rings have Ako $\implies$ Oka.

Separating the Ako and Oka properties

Ex: In the Dedekind domain $\mathbb{Z}$, the monoidal family

$$\mathcal{F} = \{(4^n) \mid n \geq 0\}$$

is Oka, but not Ako since it violates (P_3): $(4)^2 \subseteq (8) \subseteq (4)$ but $(8) \notin \mathcal{F}$.

While it seems "clear" that Ako $\not\Rightarrow$ Oka, all "natural" Ako families seem to be Oka, and several classes of rings have Ako $\implies$ Oka. Nevertheless:

There is a 2-dim. valuation domain with an Ako family that is not Oka.

Separating the Ako and Oka properties

Ex: In the Dedekind domain $\mathbb{Z}$, the monoidal family

$$\mathcal{F} = \{(4^n) \mid n \geq 0\}$$

is Oka, but not Ako since it violates (P_3): $(4)^2 \subseteq (8) \subseteq (4)$ but $(8) \notin \mathcal{F}$.

While it seems "clear" that Ako $\not\Rightarrow$ Oka, all "natural" Ako families seem to be Oka, and several classes of rings have Ako $\implies$ Oka. Nevertheless:

There is a 2-dim. valuation domain with an Ako family that is not Oka.

It should be easier to find these!

Question: Is there an example of a noetherian domain with an Ako family that is not Oka?

1 When “maximal implies prime” in commutative algebra

2 When “maximal implies prime” for one-sided ideals

3 A two-sided Prime Ideal Principle

One-sided primes

The theory for commutative rings was so nice, I wondered:

Question: What can be said about a right ideal in a noncommutative ring that is maximal with respect to not being finitely generated? Is it "prime" in some suitable sense?

One-sided primes

The theory for commutative rings was so nice, I wondered:

Question: What can be said about a right ideal in a noncommutative ring that is maximal with respect to not being finitely generated? Is it "prime" in some suitable sense?

There are several existing notions of "prime right ideals" in noncommutative algebra, but none of them answer the question above.

Thus, we needed a *new notion of prime right ideals*.

Completely prime right ideals

Def: A proper right ideal $P \subsetneq R$ is completely prime if, for all $a, b \in R$,

$$(aP \subseteq P \text{ and } ab \in P) \implies a \in P \text{ or } b \in P.$$

Completely prime right ideals

Def: A proper right ideal $P \subsetneq R$ is completely prime if, for all $a, b \in R$,

$$(aP \subseteq P \text{ and } ab \in P) \implies a \in P \text{ or } b \in P.$$

Notice immediately:

- A two-sided ideal $P \trianglelefteq R$ is completely prime as a right ideal if and only if it is a completely prime ideal: R/P has no zero-divisors. (These are rare in general noncommutative rings.)

Completely prime right ideals

Def: A proper right ideal $P \subsetneq R$ is completely prime if, for all $a, b \in R$,

$$(aP \subseteq P \text{ and } ab \in P) \implies a \in P \text{ or } b \in P.$$

Notice immediately:

- A two-sided ideal $P \trianglelefteq R$ is completely prime as a right ideal if and only if it is a completely prime ideal: R/P has no zero-divisors. (These are rare in general noncommutative rings.)
- The completely prime right ideals of a commutative ring R are the same as the prime ideals of R.

Completely prime right ideals

Def: A proper right ideal $P \subsetneq R$ is completely prime if, for all $a, b \in R$,

$$(aP \subseteq P \text{ and } ab \in P) \implies a \in P \text{ or } b \in P.$$

Notice immediately:

- A two-sided ideal $P \trianglelefteq R$ is completely prime as a right ideal if and only if it is a completely prime ideal: R/P has no zero-divisors. (These are rare in general noncommutative rings.)
- The completely prime right ideals of a commutative ring R are the same as the prime ideals of R.

Module-theoretically: $P_R \subsetneq R$ is completely prime if and only if every nonzero endomorphism of $(R/P)_R$ is injective.

Too many primes? Too few?

P is completely prime $\iff$ every $0 \neq \phi \in \mathrm{End}_R(R/P)$ is injective.

This shows that rings have "enough" completely prime right ideals.

Proposition: Every maximal right ideal of a nonzero ring is completely prime.

Proof: Schur's lemma!

Too many primes? Too few?

P is completely prime $\iff$ every $0 \neq \phi \in \mathrm{End}_R(R/P)$ is injective.

This shows that rings have “enough” completely prime right ideals.

Proposition: Every maximal right ideal of a nonzero ring is completely prime.

Proof: Schur’s lemma!

On the other hand, there aren’t “too many” completely primes, except in the trivial case:

Proposition: If R is a ring in which every proper right ideal is completely prime, then R is a division ring.

Right Oka families in noncommutative rings

Given $a \in R$, $I_R \subseteq R$, denote $a^{-1}I = \{r \in R : ar \in I\}$.
If R is commutative, then $a^{-1}I = (I : a)$.

Right Oka families in noncommutative rings

Given $a \in R$, $I_R \subseteq R$, denote $a^{-1}I = \{r \in R : ar \in I\}$.
If R is commutative, then $a^{-1}I = (I : a)$.

Definition: A family $\mathcal{F}$ of right ideals in a ring R is an Oka family of right ideals (or a right Oka family) if:

1. $R \in \mathcal{F}$, and
2. For all $I_R \subseteq R$ and $a \in R$,
$$I + aR,\ a^{-1}I \in \mathcal{F} \implies I \in \mathcal{F}.$$

Right Oka families in noncommutative rings

Given $a \in R$, $I_R \subseteq R$, denote $a^{-1}I = \{r \in R : ar \in I\}$.
If R is commutative, then $a^{-1}I = (I : a)$.

Definition: A family $\mathcal{F}$ of right ideals in a ring R is an Oka family of right ideals (or a right Oka family) if:

1. $R \in \mathcal{F}$, and
2. For all $I_R \subseteq R$ and $a \in R$,
$$I + aR,\ a^{-1}I \in \mathcal{F} \implies I \in \mathcal{F}.$$

The proof of the PIP directly generalizes to:

Completely Prime Ideal Principle (CPIP) [R., '10]: Let $\mathcal{F}$ be an Oka family of right ideals in a ring R. Then any right ideal $P \in \mathrm{Max}(\mathcal{F}')$ is completely prime.

Right Oka families from cyclic module classes

As with commutative rings, we'd like a correspondence between right Oka families and classes of cyclic modules.

Problem: $R/I \cong R/J$ does not necessarily imply that $I = J$! Can we reasonably assign *any* family of cyclic modules?

Def: I and J are similar if $R/I \cong R/J$.

Right Oka families from cyclic module classes

As with commutative rings, we'd like a correspondence between right Oka families and classes of cyclic modules.

Problem: $R/I \cong R/J$ does not necessarily imply that $I = J$! Can we reasonably assign *any* family of cyclic modules?

Def: I and J are similar if $R/I \cong R/J$.

Lemma: Every right Oka family is closed under similarity: $I \in \mathcal{F}$ and $R/I \cong R/J \implies J \in \mathcal{F}$.

Right Oka families from cyclic module classes

As with commutative rings, we'd like a correspondence between right Oka families and classes of cyclic modules.

Problem: $R/I \cong R/J$ does not necessarily imply that $I = J$! Can we reasonably assign *any* family of cyclic modules?

Def: I and J are similar if $R/I \cong R/J$.

Lemma: Every right Oka family is closed under similarity: $I \in \mathcal{F}$ and $R/I \cong R/J \implies J \in \mathcal{F}$.

This makes it sensible assign $\mathcal{F} \mapsto \mathcal{C}_{\mathcal{F}} = \{M_R \mid M \cong R/I \text{ for some } I \in \mathcal{F}\}$.

Theorem: A family $\mathcal{F}$ of right ideals in R is right Oka if and only if $\mathcal{C}_{\mathcal{F}}$ is closed under cyclic extensions.

Examples of right Oka families

Examples

In any ring R, the following are right Oka families:

- The finitely generated right ideals;
- The direct summands of R_R;
- The projective right ideals;
- The right ideals generated by $< \alpha$ elements for any infinite cardinal α;
- Any right Gabriel filter;
- For a module M_R, the family of right ideals I such that every homomorphism $f\colon I \to M$ extends to some $\tilde{f}\colon R \to M$.

In particular, a right ideal maximal in the complement of any of these families is completely prime.

Applications of the CPIP

Applying the CPIP and Zorn to $\mathcal{F} = \{\text{f.g. right ideals}\}$, we obtain:

Noncommutative Cohen's Theorem [R., '10]: A ring is right noetherian if and only if all of its completely prime right ideals are finitely generated.

Applications of the CPIP

Applying the CPIP and Zorn to $\mathcal{F} = \{\text{f.g. right ideals}\}$, we obtain:

Noncommutative Cohen's Theorem [R., '10]: A ring is right noetherian if and only if all of its completely prime right ideals are finitely generated.

With suitable choices of $\mathcal{F}$, we also find:

Theorem [R. '10]: Let R be a ring.

- A maximal point annihilator $\text{ann}(m)$ of a module $M_R \neq 0$ is completely prime.
- R is a domain iff every completely prime right ideal of R contains a left regular element ($\text{ann}_\ell(s) = 0$).

The "Cohen-Kaplansky" Theorem

Recall one of Kaplansky's results: a commutative ring is a principal ideal ring iff every prime ideal is principal.

We can show:

Theorem [R., '12]: A ring is a principal right ideal ring (PRIR) iff every completely prime ideal of R is principal.

The "Cohen-Kaplansky" Theorem

Recall one of Kaplansky's results: a commutative ring is a principal ideal ring iff every prime ideal is principal.

We can show:

Theorem [R., '12]: A ring is a principal right ideal ring (PRIR) iff every completely prime ideal of R is principal.

Expected proof:
For a ring R, let $\mathcal{F}_{\text{pr}} = \{\text{principal right ideals of } R\}$. Then $\mathcal{F}_{\text{pr}}$ is a right Oka family.

The "Cohen-Kaplansky" Theorem

Recall one of Kaplansky's results: a commutative ring is a principal ideal ring iff every prime ideal is principal.

We can show:

Theorem [R., '12]: A ring is a principal right ideal ring (PRIR) iff every completely prime ideal of R is principal.

Expected proof:
For a ring R, let $\mathcal{F}_{\text{pr}} = \{\text{principal right ideals of } R\}$. Then $\mathcal{F}_{\text{pr}}$ is a right Oka family. . . Or is it?

Families of principal right ideals

Recall that right Oka families must be closed under similarity.

Example

In the first Weyl $R = k\langle x, y \mid xy = yx + 1\rangle$, the family $\mathcal{F}_{\mathrm{pr}}$ is not closed under similarity, and therefore not right Oka, because

$$R/xR \cong R/(x^2 R + (1 + xy)R).$$

How to solve the problem: take $\mathcal{F}_{\mathrm{pr}}^{\circ} \subseteq \mathcal{F}_{\mathrm{pr}}$ to be the largest subfamily that is closed under similarity.

Theorem: In every ring R, the family $\mathcal{F}_{\mathrm{pr}}^{\circ}$ is right Oka.

Applying the CPIP and Zorn to this family yields the desired proof.

Kaplansky's theorem for noetherian rings

Recall the other part of Kaplansky's result: a commutative noetherian ring is a principal ideal ring iff all of its maximal ideals are principal.

This can be generalized to:

Theorem [R., '12]: A (left and right) noetherian ring is a principal right ideal ring iff all of its maximal right ideals are principal.

Kaplansky's theorem for noetherian rings

Recall the other part of Kaplansky's result: a commutative noetherian ring is a principal ideal ring iff all of its maximal ideals are principal.

This can be generalized to:

Theorem [R., '12]: A (left and right) noetherian ring is a principal right ideal ring iff all of its maximal right ideals are principal.

Idea: Nontrivial structure theory yields $R = (\text{artinian}) \oplus (\text{semiprime})$. Use some mild factorization theory in the semiprime case (where atomic factorizations correspond to extensions of simple modules), argue $\text{r.Kdim}(R) = 1$, and finally apply the CPIP and Zorn!

Kaplansky's theorem for noetherian rings

Recall the other part of Kaplansky's result: a commutative noetherian ring is a principal ideal ring iff all of its maximal ideals are principal.

This can be generalized to:

Theorem [R., '12]: A (left and right) noetherian ring is a principal right ideal ring iff all of its maximal right ideals are principal.

Idea: Nontrivial structure theory yields $R = \text{(artinian)} \oplus \text{(semiprime)}$. Use some mild factorization theory in the semiprime case (where atomic factorizations correspond to extensions of simple modules), argue $\text{r.Kdim}(R) = 1$, and finally apply the CPIP and Zorn!

Naturally, I wondered if either of the left or right noetherian assumptions can be lifted...

Counterexamples and questions

We cannot omit the left noetherian hypothesis in general:

Example

Let $k = \mathbb{Q}(t_1, t_2, \dots)$ and $A = k[x]_{(x)}$, and fix an iso. $\theta \colon k(x) \xrightarrow{\sim} k$. Then

$$R = \{\text{power series with zero linear term}\} \subseteq A[[y;\theta]].$$

is a local right (but not left) noetherian domain whose unique maximal right ideal $\mathfrak{m} = xR$ is principal, but which is not a PRIR: the right $I = y^2 A[[y;\theta]] = y^2 R + y^3 R$ is not principal

Counterexamples and questions

We cannot omit the left noetherian hypothesis in general:

Example

Let $k = \mathbb{Q}(t_1, t_2, \dots)$ and $A = k[x]_{(x)}$, and fix an iso. $\theta\colon k(x) \xrightarrow{\sim} k$. Then

$$R = \{\text{power series with zero linear term}\} \subseteq A[[y;\theta]].$$

is a local right (but not left) noetherian domain whose unique maximal right ideal $\mathfrak{m} = xR$ is principal, but which is not a PRIR: the right $I = y^2A[[y;\theta]] = y^2R + y^3R$ is not principal

Strangely, I do not know if we can omit the right noetherian hypothesis!

Q: Does there exist a left (but not right) noetherian ring R with all maximal right ideals principal, but which is not a principal right ideal ring?

Counterexamples and questions

I have also wondered:

Q: If R is noetherian with every maximal right ideal similar to a principal right ideal, then is every right ideal of R similar to a principal right ideal?

Alternate formulation (assume noetherian): simple right modules cyclically presented $\implies$ all cyclic right modules cyclically presented?

Counterexamples and questions

I have also wondered:

Q: If R is noetherian with every maximal right ideal similar to a principal right ideal, then is every right ideal of R similar to a principal right ideal?

Alternate formulation (assume noetherian): simple right modules cyclically presented $\implies$ all cyclic right modules cyclically presented?

Counterexample: D. Smertnig has found an order in a quaternion algebra over which all right simples c.p., but not all cyclic right modules are c.p.

Counterexamples and questions

I have also wondered:

Q: If R is noetherian with every maximal right ideal similar to a principal right ideal, then is every right ideal of R similar to a principal right ideal?

Alternate formulation (assume noetherian): simple right modules cyclically presented $\implies$ all cyclic right modules cyclically presented?

Counterexample: D. Smertnig has found an order in a quaternion algebra over which all right simples c.p., but not all cyclic right modules are c.p.

Revised Q: What can we say about the structure of right noetherian rings with all simple right modules cyclically presented?

A bit of progress with D. Smertnig: if R is semiprime, then all essential right ideals are stably free.

Generalized noetherian properties

A more recent application, joint with Zehra Bilgin and Ünsal Tekir.

Let $S \subseteq R$ be a multiplicative set. Straightforward generalizations from commutative notions due to D. D. Anderson and T. Dumitrescu (2002):

Def: A module M_R is:

- S-finite if there exists f.g. $F_R \subseteq M$ and $s \in S$ such that $Ms \subseteq F$
- S-noetherian if every submodule is S-finite

We say R is right S-noetherian if the module R_R is.

Anderson-Dumitrescu proved an "S-version" of Cohen's Theorem: if every prime ideal of a commutative ring is S-finite, then the ring is S-noetherian.

The CPIP and the S-noetherian property

Lemma: For any multiplicative subset S of a ring R, the family of S-finite right ideals is right Oka.

The lemma, CPIP, and Zorn yield a noncommutative "S-Cohen" theorem:

Theorem [Bilgin, R., Tekir '18]: A ring R is right S-noetherian if and only if every completely prime right ideal is S-finite.

The CPIP and the S-noetherian property

Lemma: For any multiplicative subset S of a ring R, the family of S-finite right ideals is right Oka.

The lemma, CPIP, and Zorn yield a noncommutative "S-Cohen" theorem:

Theorem [Bilgin, R., Tekir '18]: A ring R is right S-noetherian if and only if every completely prime right ideal is S-finite.

Another CPIP application yields "associated primes" of certain S-noetherian modules.

Theorem [BRT '18]: For a multiplicative set S in a ring R, suppose that M_R is right S-noetherian. If M is S-torsionfree, then there exists $m \in M$ such that $P = \mathrm{ann}(m)$ is a completely prime right ideal.

Understanding the S-noetherian property

Anderson & Dumitrescu characterized commutative R as S-noetherian if and only if $S^{-1}R$ is noetherian and the f.g. ideals of R have a special "S-saturation" property.

But in the noncommutative case, I don't know:

Question: Under what conditions should we expect R and S to produce a right S-noetherian ring?

Must S be a (right) Ore set? If we assume S is Ore, can we answer the above?

Understanding the S-noetherian property

One may characterize S-noetherian modules with suitable "S-analogues" of the usual ACC and maximality properties, adapted from similar notions over commutative rings due to Ahmed and Sana (2016).

Theorem [Bilgin, R., Tekir '18]: Let S be a multiplicative subset of a ring R, and M_R a module. The following are equivalent:

1. M is S-noetherian;
2. Every nonempty chain $\{N_i\}_{i \in I}$ of submodules of M is S-stationary: there exist $j \in I$ and $s \in S$ such that $N_i s \subseteq N_j$ for all i;
3. Every nonempty set $\mathcal{F}$ of submodules of M has an S-maximal element: there exists $s \in S$ such that, if $L \in \mathcal{F}$ with $M \subseteq L$, then $Ls \subseteq M$.

This appears to be new even for commutative rings.

Prime ideals in noncommutative rings

Unlike the situation with right ideals, there is a standard definition of prime *two-sided* ideals in noncommutative ring theory:

Def: An proper ideal $P \subsetneq R$ is prime if it satisfies the following equivalent conditions:

- For all ideals $A, B \trianglelefteq R$, $AB \subseteq P \implies A \in P$ or $B \in P$;
- For all $a, b \in R$, $aRb \subseteq P \implies a \in P$ or $b \in P$.

For commutative R, this is equivalent to the usual definition.

Every maximal ideal is prime, so we expect to have a reasonable supply of prime ideals in a general noncommutative ring.

A PIP for two-sided ideals?

There are (fewer, but still) some known "maximal implies prime" results for noncommutative rings in the literature. Should we expect a Prime Ideal Principle in this context?

It proved more difficult to find a good notion of Oka families for two-sided ideals. Notably, there is no corresponding version of Cohen's theorem for two-sided ideals in a noncommutative ring.

Ex: For a field k, the ring $R = \begin{pmatrix} k & k[x] \\ 0 & k \end{pmatrix}$ has two primes that are both finitely generated, but an infinitely generated nilradical.

Given the lack of rich, guiding examples, here is the best I could do...

A Prime Ideal Principle for two-sided ideals

Notation: For ideals $I, J \trianglelefteq R$, we denote the following ideals:

$$IJ^{-1} = \{r \in R \mid rJ \subseteq I\} \quad \text{and} \quad J^{-1}I = \{r \in R \mid Jr \subseteq I\}.$$

Def: A family $\mathcal{F}$ of two-sided ideals in a ring R is an Oka family if

1. The ideal $R \in \mathcal{F}$, and
2. For all $I \trianglelefteq R$ and $a \in R$,
$$I + (a), I(a)^{-1}, (a)^{-1}I \in \mathcal{F} \implies I \in \mathcal{F}.$$

A Prime Ideal Principle for two-sided ideals

Notation: For ideals $I, J \trianglelefteq R$, we denote the following ideals:

$$IJ^{-1} = \{r \in R \mid rJ \subseteq I\} \quad \text{and} \quad J^{-1}I = \{r \in R \mid Jr \subseteq I\}.$$

Def: A family $\mathcal{F}$ of two-sided ideals in a ring R is an Oka family if

1. The ideal $R \in \mathcal{F}$, and
2. For all $I \trianglelefteq R$ and $a \in R$,

$$I + (a), I(a)^{-1}, (a)^{-1}I \in \mathcal{F} \implies I \in \mathcal{F}.$$

Note: Both Oka families of two-sided ideals and right Oka families restrict to the same old Oka families in the case where R is commutative.

A Prime Ideal Principle for two-sided ideals

Notation: For ideals $I, J \trianglelefteq R$, we denote the following ideals:

$$IJ^{-1} = \{r \in R \mid rJ \subseteq I\} \quad \text{and} \quad J^{-1}I = \{r \in R \mid Jr \subseteq I\}.$$

Def: A family $\mathcal{F}$ of two-sided ideals in a ring R is an Oka family if

1. The ideal $R \in \mathcal{F}$, and
2. For all $I \trianglelefteq R$ and $a \in R$,
$$I + (a), I(a)^{-1}, (a)^{-1}I \in \mathcal{F} \implies I \in \mathcal{F}.$$

Note: Both Oka families of two-sided ideals and right Oka families restrict to the same old Oka families in the case where R is commutative.

Prime Ideal Principle [R. '16]: Let $\mathcal{F}$ be an Oka family of two-sided ideals in a ring R. Then any ideal $P \in \mathrm{Max}(\mathcal{F}')$ is prime.

Proof of (another) Prime Ideal Principle

Prime Ideal Principle [R. '16]: Let $\mathcal{F}$ be an Oka family of two-sided ideals in a ring R. Then any ideal $P \in \mathrm{Max}(\mathcal{F}')$ is prime.

This requires the unusual, but elementary:

Lemma: If a proper ideal P of a ring R is not prime, then there exist $a, b \in R \setminus P$ such that $aRb \subseteq P$ and $bRa \subseteq P$.

Proof of (another) Prime Ideal Principle

Prime Ideal Principle [R. '16]: Let $\mathcal{F}$ be an Oka family of two-sided ideals in a ring R. Then any ideal $P \in \mathrm{Max}(\mathcal{F}')$ is prime.

This requires the unusual, but elementary:

Lemma: If a proper ideal P of a ring R is not prime, then there exist $a, b \in R \setminus P$ such that $aRb \subseteq P$ and $bRa \subseteq P$.

Proof of PIP: Let $P \in \mathrm{Max}(\mathcal{F}')$, and suppose toward a contradiction that P were not prime. The lemma yields $a, b \in R \setminus P$ such that $aRb, bRa \subseteq P$. Note that $P(a)^{-1}$ and $(a)^{-1}P$ contain both P and $b \notin P$. Maximality of P yields that

$$P + (a), P(a)^{-1}, (a)^{-1}P \in \mathcal{F},$$

and the Oka property now gives the contradiction $P \in \mathcal{F}$. $\square$

Some two-sided Oka families

Examples

For a ring R, the following families of ideals are Oka:

- Ideals I intersecting a given multiplicative set (or m-system) S
- Ideals I such that R/I satisfies a polynomial identity
- Ideals I such that R/I is Dedekind-finite

So an ideal maximal *outside* any of these families is prime.

Some two-sided Oka families

Examples

For a ring R, the following families of ideals are Oka:

- Ideals I intersecting a given multiplicative set (or m-system) S
- Ideals I such that R/I satisfies a polynomial identity
- Ideals I such that R/I is Dedekind-finite

So an ideal maximal *outside* any of these families is prime.

Ex: If R is a finitely generated k-algebra, then the ideals of finite codimension form an Oka family.

This can be used to recover a result of Small via the PIP:

Theorem [Small]: If an infinite-dimensional, finitely generated k-algebra R is just infinite (i.e., every proper homomorphic image is finite-dimensional), then R is prime.

A final question

Recall that for right ideals, there is a connection between right Oka families and classes of modules that are closed under extensions.

Unfortunately, I do not know of a similar correspondence between Oka families of two-sided ideals and, say, certain classes of bimodules.

Questions:

- Is there a way to view Oka families of two-sided ideals as corresponding to certain classes of bimodules?
- If not, then is there a "better" notion of two-sided Oka families that leads to a broader range of examples?
- In any case, how can we find new, unexpected examples of Oka families and "maximal implies prime" results for two-sided ideals?

Thank you! (And some references)

Z. Bilgin, M.Reyes, and Ü. Tekir, *On right S-Noetherian rings and S-Noetherian modules*, Comm. Algebra, 2018.

I. Cohen, *Commutative rings with minimum condition*, Duke Math. J., 1950.

I. Kaplansky, *Elementary divisors and modules*, Trans. AMS, 1949.

T. Y. Lam and M. Reyes, *A Prime Ideal Principle in commutative algebra*, J. Algebra, 2008.

T. Y. Lam and M. Reyes, *Oka and Ako ideal families in commutative rings*, Contemp. Math., 2009.

K. Oka, *Sur les fonctions analytiques de plusieurs variables*, J. Math. Soc. Japan, 1951.

M. Reyes, *A one-sided Prime Ideal Principle for noncommutative rings*, J. Algebra Appl., 2010.

M. Reyes, *Noncommutative generalizations of theorems of Cohen and Kaplansky*, Algebr. Represent. Theory, 2012.

M. Reyes, *A prime ideal principle for two-sided ideals*, Comm. Algebra, 2016.