A type of generalized factorization on domains

τ-Factorizations

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Outline

Notation

- D denotes an integral domain
- $D^\sharp$ is the set of nonzero nonunits elements of D
- τ denotes a symmetric relation on $D^\sharp$

Outline

1. Notation and Definitions
   - Definitions
   - Relations

2. Equivalence relations
   - Motivation
   - Some results (Ortiz and Serna)

Definition of a τ-Factorization

Definition

We say $x \in D^\sharp$ has **a τ-factorization** if $x = \lambda x_1 \cdots x_n$ where λ is a unit in D and $x_i \tau x_j$ for each $i \neq j$.

- We say x is **a τ-product** of $x_i \in D^\sharp$ and each x_i is a **τ-factor** of x (we write $x_i \mid_\tau x$).
- Vacuously, $x = x$ and $x = \lambda \cdot (\lambda^{-1} x)$ are τ-factorizations, known as the trivial ones.

Definition

In general, $x \mid_\tau y$ (read x **τ-divides** y) means y has a τ-factorization with x as a τ-factor.

Definition

We call $x \in D^\sharp$ a **τ-atom,** if the only τ-factorizations of x are of the form $\lambda(\lambda^{-1}x)$ (the **trivial τ-factorizations**).

- Example: Irreducible elements are τ-atoms (for any relation τ on $D^\sharp$).

Definition

A τ-factorization $\lambda x_1 \cdots x_n$ is **a τ-atomic factorization** if each x_i is a τ-atom.

Definition

If you exchange the factorization, irreducible elments and divide operator by τ-factorization, τ-atom and $|_\tau$ operator in the definitions of GCD domain, UFD, HFD, FFD, BFD and ACCP. We obtain the notions of:

- τ-GCD domain,
- τ-UFD,
- τ-HFD,
- τ-FFD,
- τ-BFD, and
- τ-ACCP.

Examples

Example

Let $\tau_D = D^\sharp \times D^\sharp$ (the greatest relation), then the τ_D-factorizations are the usual factorizations.

Example

Let $\tau_\emptyset = \varnothing$ (the trivial), then we have that every element is a τ-atom. So any integral domain is in fact a $\tau_\emptyset$-UFD.

Example

Let $\tau_S = S \times S$, where $S \subset D^\sharp$. Hence you can consider S is the set of primes, irreducible, primals, primary elements, rigid elements,...

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Types of Relations

- Associate preserving
- Divisive
- Multiplicative

Definitions and Properties

Let $x, y, z \in D^\sharp$.

- We say τ is **associate-preserving** if $x\tau y$ and $y \sim z$ implies $x\tau z$.
- If $\lambda x_1 \cdots x_n$ is a τ-fatorization, then $x_1 \cdots x_{i-1} \cdot (\lambda x_i) \cdot x_{i+1} \cdots x_n$ is also a τ-factotization.

Types of Relations

- Associate preserving
- Divisive
- Multiplicative

Definitions and Properties

Let $x, y, z \in D^\sharp$.

- We say τ is **divisive** if $x\tau y$ and $z \mid x$ implies $z\tau y$.
- Divisive implies associate-preserving.
- If τ is divisive, then we can do τ-refinements.
- That is, if $x_1 \cdots x_n$ is a τ-factorization and $z_1 \cdots z_m$ is a τ-factorization of x_i then $x_1 \cdots x_{i-1} \cdot z_1 \cdots z_m \cdot x_{i+1} \cdots x_n$ is also a τ-factorization.

Types of Relations

- Associate preserving
- Divisive
- Multiplicative

Definitions and Properties

Let $x, y, z \in D^\sharp$.

- We say τ is **multiplicative** if $x\tau y$ and $x\tau z$ implies $x\tau(yz)$.
- If τ is multiplicative, then each nontrivial τ-factorization can be written into a τ-product of length 2

More Examples

Example

Let $\tau_{(n)} = \{(a,b) \mid a-b \in (n)\}$ a relation on $\mathbb{Z}^\sharp$, for each $n \geq 0$.

- For $n=1$, we obtained the usual factorizations.
- Note that for $n \geq 2$, $\tau_{(n)}$ is never divisive, but it is multiplicative and associate-preserving for $n=2$.
- (Hamon) $\mathbb{Z}$ is a $\tau_{(n)}$-UFD if and only if $n = 0,1$.
- (Hamon and Juett) $\mathbb{Z}$ is a $\tau_{(n)}$-atomic domain if and only if $n = 0,1,2,3,4,5,6,8,10$.
- (Ortiz) $\mathbb{Z}$ is a $\tau_{(n)}$-GCD domain if and only if is a $\tau_{(n)}$-UFD.

More Examples

Example

Let $*$ be a start-operation on D. Then define $x\tau_* y \iff (x,y)^* = D$, that is, x and y are $*$-comaximal.

- It is both multiplicative and divisive.
- If $\star = d$, then a d-factorization is the known comaximal factorization defined by McAdam and Swam.

Diagram of Properties (Anderson and Frazier)

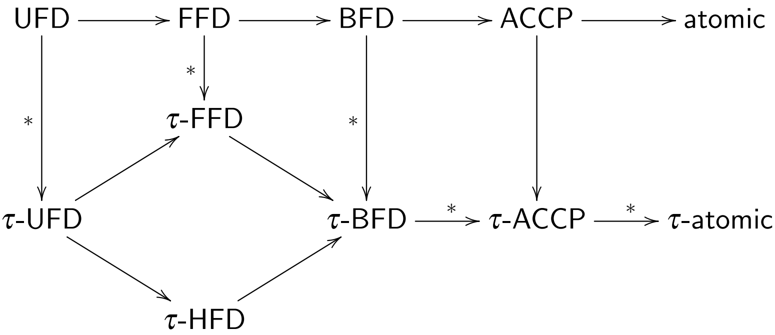

Figure: Diagram of structures and τ-structures, when τ is divisive.

Definition of "$\leq$"

Definition

We say $\tau_1 \leq \tau_2$, if $\tau_1 \subseteq \tau_2$ as sets.

Theorem

Let D be an integral domain and τ_1,τ_2 be two relations on $D^\sharp$. The following are equivalent:

- $\tau_1 \leq \tau_2$.
- *For any $x, y \in D^\sharp$, $x\tau_1 y \Rightarrow x\tau_2 y$.*
- *Any τ_1-factorization is a τ_2-factorization.*

Theorem (Ortiz)

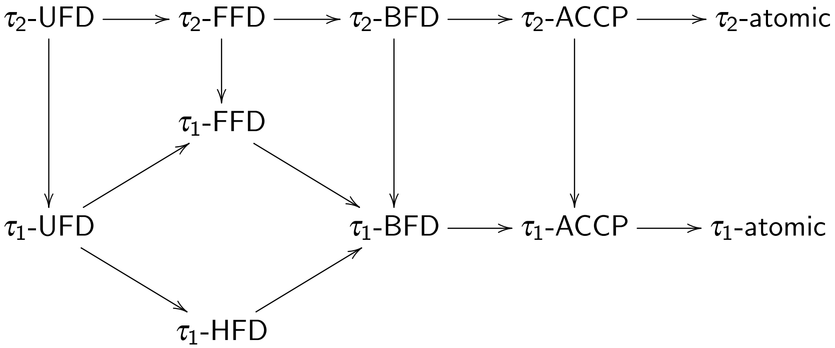

Figure: Properties when $\tau_1 \subseteq \tau_2$ both divisive and τ_2 multiplicative

Theorem(Juett)

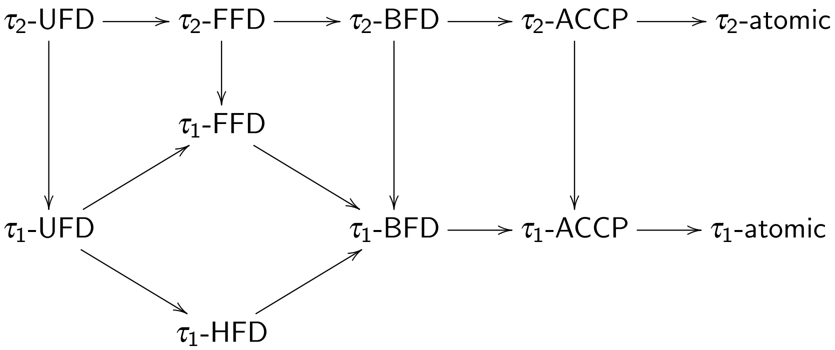

Figure: Properties when $\tau_1 \subseteq \tau_2$, τ_1 divisive and τ_2 refinable and associated-preserving

Outline

- Equivalence relations have historical precedent.
- Equivalence relation are less artificial relations.
- There is only one divisive equivalence relation τ_D.
- Divisive seems to be more-less understood to be good type of relation.

Outline

Diagram of Properties (Ortiz and Serna)

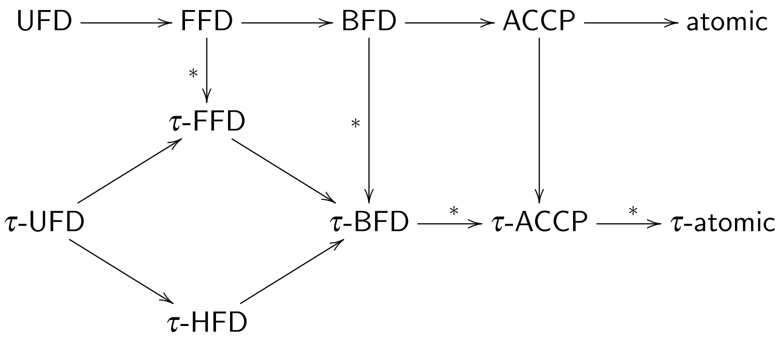

Figure: In this case τ is an associated-preserving multiplicative equivalence relation.

Associated-preserving clousure of an equivalence relation

Definition

Let τ be an equivalence relation on $D^\sharp$. The associated-preserving clousure of τ is denoted by τ', which is the intersection of all associated-preserving equivalence relations on $D^\sharp$ containing τ.

Theorem

Suppose τ (is unital) has the following property: for any $x, y \in D^\sharp$ and $\lambda \in U(D)$, if $x\tau y$, then $(\lambda x)\tau(\lambda y)$. Then

$$\tau' = \{(\mu_1 x, \mu_2 y) | (x, y) \in \tau \text{ and } \mu_1, \mu_2 \in U(D)\}$$

Theorem

If τ is unital equivalence relation, then we may assume τ is associated-preserving, because

- x *has a* τ'*-factorization if and only if* x *has a* τ*-factorization*
- $x \mid_{\tau'} y$ *if and only if* $x \mid \tau y$
- x *is a* τ'*-atom if and only if* x *is a* τ*-atom*
- D *is* τ'*-atomic if and only if* D *is* τ*-atomic*

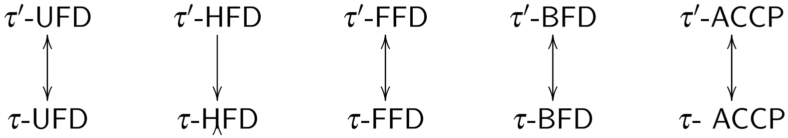

Figure: Properties of the associated-preserving equivalence unital relation

Example

- If you consider $\tau'_{(n)}$, then
- $\tau'_{(n)}$ is an equivalence relation with the "half" number of equivalence classes than $\tau_{(n)}$,
- $\tau'_{(n)}$ is always associated-preserving relation,
- $\tau'_{(n)}$ is multiplicative only if $n \in \{1, 2, 3, 6\}$ (it is also the multiplicative clousure), and
- $\tau'_{(n)}$ coincides with Lanterman relation called $\mu_{(n)}$, presented at JMM 2013.
- Something about notions of $\tau_{(n)}$-number theory.

- Thanks for the invitation.
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