Non-unique factorizations in rings of integer-valued polynomials on certain Dedekind domains

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Outline

- Preliminaries on $\mathrm{Int}(D)$ and factorizations
- What is known in $\mathrm{Int}(\mathbb{Z})$
- New results

$\mathrm{Int}(D)$

Definition 1

Let D be a domain and $K = q(D)$.

$$\mathrm{Int}(D) = \{f \in K[x] \mid \forall\ a \in D, f(a) \in D\} \subseteq K[x]$$

Remark 1

1. For all $f \in K[x]$, $f = \frac{g}{b}$ where $g \in D[x]$ and $b \in D \setminus \{0\}$.
2. $f = \frac{g}{b}$ is in $\mathrm{Int}(D)$ if and only if $b \mid g(a)$ for all $a \in D$.

Examples

1. $D[x] \subseteq \mathrm{Int}(D)$
2. $\binom{x}{n} = \frac{x(x-1)(x-2)\cdots(x-n+1)}{n!} \in \mathrm{Int}(\mathbb{Z})$

Int(D) cont'd

- Int(D) is in general not a UFD e.g., in Int($\mathbb{Z}$)

$$
\begin{aligned}
\frac{x(x-1)(x-2)}{2} &= \frac{x(x-1)}{2}\,(x-2) \\
&= x\,\frac{(x-1)(x-2)}{2} \\
&= 3\,\frac{x(x-1)(x-2)}{6}
\end{aligned}
$$

Factorization terms

Definition 2

Let $0 \neq r \notin R^{\times}$.

1. Two factorizations of

$$r = r_1 \cdots r_n = s_1 \cdots s_m$$

   are called essentially the same if $n = m$ and, after some possible reordering, $r_j \sim s_j$ for $1 \leq j \leq m$. Otherwise, the factorizations are called essentially different.

2. The set of lengths of r is

$$L(r) = \{n \in \mathbb{N} \mid r = r_1 \cdots r_n\}$$

   where $r_1, \ldots, r_n$ are irreducibles.

What is known in $\mathrm{Int}(\mathbb{Z})$

Theorem 1 (Frisch, 2013)

Let $1 \leq m_1 \leq m_2 \leq \cdots \leq m_n \in \mathbb{N}$. Then there exists a polynomial $H \in \mathrm{Int}(\mathbb{Z})$ with n essentially different factorizations of lengths $m_1 + 1, \ldots, m_n + 1$.

Corollary 1

Every finite subset of $\mathbb{N}_{>1}$ is a set of lengths of an element of $\mathrm{Int}(\mathbb{Z})$.

(Kainrath, 1999) Corollary 1 for Krull monoids with infinite class group such that each divisor class contains a prime divisor.

What is known in Int($\mathbb{Z}$)

Proposition 1 (Frisch, 2013)

For every $n \geq 1$ there exist irreducible elements $H, G_1, \ldots, G_{n+1}$ in Int($\mathbb{Z}$) such that $xH(x) = G_1(x) \cdots G_{n+1}(x)$.

(Geroldinger & Halter-Koch, 2006)

1. If $\theta : H \longrightarrow M$ is a transfer homomorphism, then;
   - (i) $u \in H$ is irreducible in H if and only if $\theta(u)$ is irreducible in M.
   - (ii) For $u \in H$, $L(u) = L(\theta(u))$
2. If u, v are irreducibles elements of a block monoid with u fixed, then $\max L(uv) \leq |u|$, where $|u| \in \mathbb{N}_{\geq 0}$.
3. Any monoid which allows a transfer homomorphism to a block monoid must have the property in 2.

Monoids which allow transfer homomorphisms to block monoids are called transfer Krull monoids.

What is known in $\mathrm{Int}(\mathbb{Z})$

Corollary 2

$(\mathrm{Int}(\mathbb{Z}) \setminus \{0\}, \bullet)$ is not a transfer Krull monoid.

New results

Motivation question: Are there other domains D such that $\mathrm{Int}(D)$ is not a transfer Krull monoid?

If D is a Dedekind domain such that;

1. D has infinitely many maximal ideals and,
2. all the maximal ideals are of finite index.

Then $\mathrm{Int}(D)$ is not a transfer Krull monoid.

Examples of our Dedekind domains

1. $\mathbb{Z}$
2. $\mathcal{O}_K$, the ring of integers of a number field K
3. The integral closure of $\mathbb{Z}_p[x]$ in a finite dimensional extension of $\mathbb{Z}_p(x)$

New results

Let D be a Dedekind domain such that;

1. D has infinitely many maximal ideals and,
2. all the maximal ideals are of finite index.

Theorem 2

For every $n \geq 1$ there exist irreducible elements $H, G_1, \ldots, G_{n+1}$ in $\mathrm{Int}(D)$ such that $xH(x) = G_1(x) \cdots G_{n+1}(x)$.

Theorem 3

Let $1 \leq m_1 \leq m_2 \leq \cdots \leq m_n \in \mathbb{N}$. Then there exists a polynomial $H \in \mathrm{Int}(D)$ with n essentially different factorizations of lengths $m_1 + 1, \ldots, m_n + 1$.