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DIRECTED UNIONS OF LOCAL QUADRATIC AND MONOIDAL TRANSFORMS AND GCD DOMAINS

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0. Introduction.

Let $(R, \mathfrak{m})$ be a regular local ring of dimension $d \geq 2$ and let $x \in \mathfrak{m} \backslash \mathfrak{m}^2$. A local quadratic transform (LQT) of R is an overring of the form

$$R_1 = R\left[\frac{\mathfrak{m}}{x}\right]_{\mathfrak{m}_1}$$

where $\mathfrak{m}_1$ is a maximal ideal of $R[\frac{\mathfrak{m}}{x}]$ lying over $\mathfrak{m}$.

R_1 is a regular local ring of dimension

$$d - \text{trdeg}_{\frac{R}{\mathfrak{m}}}\left(\frac{R_1}{\mathfrak{m}_1}\right)$$

and its ideal $\mathfrak{m}R_1 = xR_1$ is principal.

Let V be a valuation overring of R such that $\mathfrak{m}_V \cap R = \mathfrak{m}$. There exists a unique sequence of regular local rings

$$R = R_0 \subseteq R_1 \subseteq \ldots \subseteq R_n \subseteq R_{n+1} \subseteq \ldots \subseteq V$$

such that R_{n+1} is a LQT of R_n, $\forall n$.

Theorem(Abyhankar '56)

The following assertions are equivalent:

(1) $\exists n$ such that $R_n = R_{n+1} = \ldots = V$.

(2) V is the order valuation ring of R_{n-1}.

(3) V is a prime divisor of R (i.e. $\text{trdeg}_{\frac{R}{\mathfrak{m}}}\left(\frac{V}{\mathfrak{m}_V}\right) = d - 1$).

Assume $\text{trdeg}_{\frac{R}{\mathfrak{m}}}\left(\frac{V}{\mathfrak{m}_V}\right) = 0$ and define the ring $S = \bigcup_{n\geq 0}^{\infty} R_n \subseteq V$. S is called a quadratic Shannon Extension of R and it is a local domain with maximal ideal $\mathfrak{m}_S = \bigcup_{n\geq 0}^{\infty} \mathfrak{m}_n$.

Theorem(Abyhankar '56)
If $d = 2$, then $S = V$.

Example(Shannon '73)
Let $d = 3$ and $\mathfrak{m} = (x, y, z)R$.

(1) Define for $n \geq 1$, $R_n = R_{n-1}\left[\frac{y}{x^n}, \frac{z}{x^n}\right]_{(x, \frac{y}{x^n}, \frac{z}{x^n})}$ and $S = \bigcup_{n\geq 0}^{\infty} R_n$. S is a non archimedean domain of dimension 3 with principal maximal ideal $\mathfrak{m}_S = xS$. Since $\frac{y}{z}, \frac{z}{y} \notin S$, S is not a valuation ring.

(2) Let V be a rank 1 valuation overring of R such that $v(z) > v(x) + v(y)$. Then the Shannon extension S along V is an archimedean domain of dimension 2 and $\mathfrak{m}_S = \mathfrak{m}_S^2$.

Shannon's examples motivated the following recent articles (2014-2017) and further research:

- Ideal theory of infinite directed unions of local quadratic transforms
  (W. Heinzer, K. A. Loper, B. Olberding, H. Schoutens and M. Toeniskoetter)

- Asymptotic properties of infinite directed unions of local quadratic transforms
  (W. Heinzer, B. Olberding and M. Toeniskoetter)

- Directed unions of local quadratic transforms of a regular local ring and pullbacks,
  (-, W. Heinzer, B. Olberding and M. Toeniskoetter)

1. Properties of quadratic Shannon extensions.

Let S be a quadratic Shannon extension of a regular local ring R. Then:

- The maximal ideal of S, $\mathfrak{m}_S$ is either principal or idempotent.

- Any non maximal prime ideal P of S is such that $S_P = (R_n)_{P \cap R_n}$ for $n >> 0$.

- For any $n >> 0$, set $R_{n+1} = R_n \left[\frac{\mathfrak{m}_n}{x_n}\right]_{\mathfrak{m}_{n+1}}$. Then $x_n S$ is an $\mathfrak{m}_S$-primary ideal.

- S is Noetherian if and only if it is a DVR.

- Let $x \in S$ be an $\mathfrak{m}_S$-primary element. The ring $T = S[\frac{1}{x}]$ is a Noetherian UFD and, when S is not a DVR, it is the minimal proper Noetherian overring of S. We call T the Noetherian hull of S.

Let V_n be the order valuation ring of R_n. We call the ring

$$V_B := \lim_{n \to \infty} V_n = \bigcup_{n \geq 0} \bigcap_{i \geq n} V_i$$

the Boundary Valuation ring of S. A useful property is that

$$S = T \cap V_B.$$

S is a valuation domain if and only if either $\dim(S) = 1$ or $\dim(S) = 2$ and the value group of V_B is $\mathbb{Z} \oplus G$ with $G \leq \mathbb{Q}$.

Let W be the rank 1 valuation overring of V_B. It is possible to characterize the complete integral closure S^* of S:

- When S is non archimedean the complete integral closure S^* of S is equal to the Noetherian hull T.

- When S is archimedean, the complete integral closure of S is

$$S^* = (\mathfrak{m}_S :_{Q(R)} \mathfrak{m}_S) = W \cap T.$$

It follows that $S = S^*$ if and only if V_B has rank 1.

2. GCD property for quadratic Shannon extensions

An integral domain D is a GCD domain if for every $a, b \in D$ the ideal $aD \cap bD$ is principal.

Theorem

Let S be a quadratic Shannon extension of a regular local ring R. The following assertions are equivalent:

(1) S is a GCD domain.

(2) S is a valuation domain.

Proof (sketch).

a) Non archimedean case:

Let S be a non archimedean Shannon extension and let $x \in S$ be an $\mathfrak{m}_S$-primary element. Then, it is proved that the ideal $Q = \bigcap_{n \geq 0} x^n S$ is a nonzero prime ideal and any non maximal prime ideal of S is contained in Q. Hence S occurs in the following pullback diagram:

$$\begin{array}{ccc} S & \twoheadrightarrow & \frac{S}{Q} \\ \downarrow & & \downarrow \\ S_Q & \twoheadrightarrow & \kappa(Q) \end{array}$$

where S/Q is a rank 1 valuation domain and S_Q is equal to the Noetherian hull T.

By a theorem of Gabelli and Houston (in Coherentlike conditions in pullbacks '97), if S is a GCD domain, S_Q is a valuation domain. Since also S/Q is a valuation domain, S itself is a valuation domain.

b) Archimedean case but not completely integrally closed:

Assume S not a valuation domain. Hence $\mathfrak{m}_S$ is not finitely generated. Take $\theta \in S^* \setminus S$. Since $S^* = (\mathfrak{m}_S :_{Q(R)} \mathfrak{m}_S)$,

$$\mathfrak{m}_S \subseteq \theta^{-1}\mathfrak{m}_S \cap S \subseteq \theta^{-1}S \cap S.$$

But $\theta \notin S$ and hence $\mathfrak{m}_S = \theta^{-1}S \cap S$ and therefore S is not a GCD domain.

c) Archimedean case and completely integrally closed:

Assume S not a valuation domain. Hence $S = T \cap W$ with W a rank 1 valuation domain.
Using properties of the Boundary valuation ring, we get that $\forall \epsilon > 0$, $\exists x \in \mathfrak{m}_S$ such that $0 < w(x) < \epsilon$.
Take $f, g \in S$ such that $fT, gT \subsetneq T$ and $fT \cap gT = fgT$. Then, assuming $w(f) \geq w(g)$, define:

$$I := fS \cap gS = (fT \cap gT) \cap \{a \in S \mid w(a) > w(f)\}.$$

Hence, given $a \in I$, it is possible to find $x \in \mathfrak{m}_S$ such that $0 < w(x) < w(\frac{a}{f})$. Thus $\frac{a}{x} \in I$ and $I = \mathfrak{m}_S I$. It follows that I is not finitely generated and S not a GCD domain.

3. Monoidal Shannon extensions.

Let $(R, \mathfrak{m})$ be a regular local ring of dimension $d \geq 3$ and let $\mathfrak{p}$ be a regular prime ideal of R (i.e. $R/\mathfrak{p}$ is a regular local ring) with $\operatorname{ht}\mathfrak{p} > 1$. Take $x \in \mathfrak{p} \setminus \mathfrak{p}^2$. A local monoidal transform of R is an overring of the form

$$R_1 = R\left[\frac{\mathfrak{p}}{x}\right]_{\mathfrak{m}_1}$$

where $\mathfrak{m}_1$ is a maximal ideal of $R[\frac{\mathfrak{p}}{x}]$ lying over $\mathfrak{m}$.

R_1 is again a regular local ring and its ideal $\mathfrak{p}R_1 = xR_1$ is principal.

Define for $n \geq 1$, $R_{n+1} = R_n[\frac{\mathfrak{p}_n}{x_n}]_{\mathfrak{m}_{n+1}}$ and, assuming $\dim(R_n) = d$ for every n, we call the union $S = \bigcup_{n \geq 0}^{\infty} R_n$ a monoidal Shannon extension of R.

Example

Let $d = 3$ and $\mathfrak{m} = (x, y, z)R$.
Let $\mathfrak{p} = (x, y)R$ and $R_1 = R\left[\frac{y}{x}\right]_{(x,z,\frac{y}{x})}$. Then let $\mathfrak{p}_1 = (z, \frac{y}{x})R_1$ and $R_2 = R_1\left[\frac{y}{xz}\right]_{(x,z,\frac{y}{xz})}$.

Following this pattern, define for $k \geq 1$,

$R_{2k} = R_{2k-1}\left[\frac{y}{x^k z^k}\right]_{(x,z,\frac{y}{x^k z^k})}$ and $R_{2k+1} = R_{2k}\left[\frac{y}{x^{k+1} z^k}\right]_{(x,z,\frac{y}{x^{k+1} z^k})}$.

$S := \bigcup_{n \geq 0}^{\infty} R_n$ is a domain of dimension 3 with maximal ideal $\mathfrak{m}_S = (x, z)S$. The ideals xS and zS are prime ideals of height 2 and the ideal $P = R_{yR} \cap S$ is a non finitely generated prime ideal of height 1. The Noetherian hull of S is the ring $T = S[\frac{1}{xz}]$.

Since $\frac{x}{z}, \frac{z}{x} \notin S$, S is not a valuation domain. But S turns out to be a GCD domain. This follows from:

Theorem

Let D be an integral domain and $x \in D$ a nonzero prime element. The following assertions are equivalent:

(1) D is a GCD domain.

(2) $D[\frac{1}{x}]$ and D_{xD} are GCD domains.

Corollary

Let D be an integral domain and $x_1, \ldots, x_n \in D$ nonzero and non associate prime elements. Call $x := x_1 \cdots x_n$.

If $D[\frac{1}{x}]$ and D_{x_iD} are GCD domains $\forall i = 1, \ldots, n$, then D is a GCD domain.

Proof of the theorem (sketch).

When D is a GCD domain, it is clear that $D[\frac{1}{x}]$ and D_{xD} are GCD domains.

Conversely take $a, b \in D$ and call $I = aD \cap bD$ and $J = (a, b)D$. If D_{xD} is a GCD domain, the already cited result of Gabelli and Houston implies that D_{xD} is a valuation domain.
Hence, without loss of generality, we may assume $a \notin xD$ (by eventually multiplying I for some element in J^{-1}). We need to show

$$I = abJ^{-1} = ab(\frac{1}{a}D \cap \frac{1}{b}D)$$

is principal.

This is equivalent to say that $K := aJ^{-1} = D \cap \frac{a}{b}D$ is principal. We get this showing that for some $y \in D$,

$$K = KD\left[\frac{1}{x}\right] \cap D = yD.$$

The first equality follows since, for $z \in KD\left[\frac{1}{x}\right] \cap D$, $zb = \frac{ad}{x^n} \in aD$ (with $d \in D$ and $n \geq 0$) since $a \notin xD$ and x is prime.
The second equality follows since $D\left[\frac{1}{x}\right]$ is a GCD domain and K is intersection of principal ideals.

Thanks for your attention!