Sets of Lengths of Puiseux Monoids

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Introduction

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Motivation: *A realization theorem for sets of lengths in numerical monoids* by Alfred Geroldinger and Wolfgang Schmid.

Main Results:

1. An atomic Puiseux monoid with full system of sets of lengths is constructed.
2. The Characterization Problem for the family of non-finitely generated Puiseux monoids is answered negatively.
3. A family of Puiseux monoids with all their sets of lengths having extremal cardinality (one or infinity) is constructed.
4. The intersection of the systems of sets of lengths of all nontrivial atomic Puiseux monoids is found.
5. We show that finding L(2) in the elementary primary Puiseux monoid is as hard as answering the Goldbach's conjecture.

Outline

1. Preliminary
2. First Main Result: A Puiseux monoid with full system of sets of lengths
3. Consequences of First Main Result
4. Second Main Result: A Puiseux monoid whose sets of lengths have extremal cardinality
5. Consequences of Second Main Result

What is a Puiseux monoid?

Definition (Puiseux monoid)

A *Puiseux monoid* is an additive submonoid of $\mathbb{Q}$ consisting of nonnegative rational numbers.

Proposition (Gilmer): Puiseux monoids account (up to isomorphism) for any possible nontrivial additive submonoid of $\mathbb{Q}$ that is not a group.

Observations:

- Not every Puiseux monoid is atomic: $\langle 1/2^n \mid n \in \mathbb{N} \rangle$.
- There are atomic Puiseux monoids containing infinitely many atoms (i.e., irreducibles): $\langle 1/p \mid p \text{ is prime} \rangle$.
- There are atomic Puiseux monoids containing non-atomic submonoids: $\langle 1/(2^p p) \mid p \text{ is prime} \rangle$ and its submonoid $\langle 1/2^n \mid n \in \mathbb{N} \rangle$.

Puiseux monoids and numerical monoids

Definition (numerical monoid)

A *numerical monoid* is a cofinite submonoid of $(\mathbb{N}_0, +)$.

Every numerical monoid is naturally a Puiseux monoid and the next characterization follows easily.

Observation: A nontrivial Puiseux monoid M is isomorphic to a numerical monoid if and only if M is finitely generated.

Remark: The family of Puiseux monoids generalizes that one of numerical monoids.

Why should we care about Puiseux monoids?

0. They are nice, beautiful, and fun.
1. They are a useful source of counter/examples to the service of commutative ring theory and factorization theory:
   - Puiseux monoids were crucial to find the first example of an atomic integral domain that fails to satisfy the ACCP (this is due to Anne Grams);
   - Anderson-Anderson-Zafrullah use Puiseux monoids to build an example of an ACCP domain that is not a bounded factorization domain (BFD).
   - Anderson-Anderson-Zafrullah use Puiseux monoids to build an example of a BFD whose integral closure is not a BFD.
2. They provide a new playground to investigate potential pathological behavior of arithmetic of factorizations as most Puiseux monoids are neither C-monoids nor transfer Krull.
3. They facilitate the study of numerical monoids by providing a common universe where infinitely many rescaled copies of numerical monoids coexist.

Some notation

Notation: Let $\mathbb{N} := \{1, 2, \dots\}$ and $\mathbb{P}_{\mathsf{fin}} := \{S \subset \mathbb{N}_{\geq 2} : |S| < \infty\}$.
Assumption: Each monoid here is assumed to be commutative, cancellative, and reduced. Let M be a monoid.

- Let $\mathcal{A}(M)$ be the set of atoms (i.e., irreducibles) of M.
- Let $\mathsf{Z}(M)$ denote the factorization monoid of M, that is the free commutative monoid on $\mathcal{A}(M)$.
- Let $\phi\colon \mathsf{Z}(M) \to M$ be the only monoid homomorphism satisfying $\phi(a) = a$ for all $a \in \mathcal{A}(M)$.
- For $x \in M$, we set $\mathsf{Z}(x) := \phi^{-1}(x)$.
- If $z = a_1 \dots a_k \in \mathsf{Z}(M)$, then $|z| := k$ is called the *length* of z.
- The *set of lengths* of $x \in M$ is $\mathsf{L}(x) := \{|z| : z \in \mathsf{Z}(x)\}$.
- The *system of sets of lengths* of M is $\mathcal{L}(M) := \{\mathsf{L}(x) : x \in M\}$.

Monoids with full system of sets of lengths

Definition (full system of sets of lengths)

A BF-monoid M is said to have *full system of sets of lengths* if $\mathcal{L}(M) = \{\{0\}, \{1\}\} \cup \mathbb{P}_{\text{fin}}$.

Theorem (Kainrath, 1999)

Let M be Krull monoid, and let G be the class group of M. If G is infinite and every class of G contains at least a prime, then M has full system of sets of lengths.

Definition: The *system of sets of lengths* of an integral domain R is $\mathcal{L}(R^\bullet)$, where $R^\bullet$ denotes the multiplicative monoid of R.

Theorem (Frisch-Nakato-Rissner, 2017)

If $\mathcal{O}_K$ is the ring of integers of a given number field K, then the domain Int$(\mathcal{O}_K)$ has full system of sets of lengths.

A realization theorem for sets of lengths

Theorem (Geroldinger-Schmid, 2017)

Let $L \subset \mathbb{N}_{\geq 2}$ be a finite nonempty set and $f : L \to \mathbb{N}$ a map. Then there exist a numerical monoid M and a squarefree element $x \in M$ such that the following conditions hold:

1. $\mathsf{L}(x) = L$;
2. $|\mathsf{Z}_\ell(x)| = f(\ell)$ *for every* $\ell \in L$.

Puiseux monoid with full system of sets of lengths

Theorem (G., 2017)

[First Main Result] There exists an atomic Puiseux monoid with full systems of sets of lengths.

Sketch of proof:

1. Number the sets in $\mathbb{P}_{\text{fin}}$, say $S_1, S_2, \dots$.
2. For each $n \in \mathbb{N}$, use Geroldinger-Schmid Theorem to find a numerical monoid $M_n \subset \mathbb{Q}_{\geq 0}$ and $x_n \in M_n$ such that $\mathsf{L}(x_n) = S_n$.
3. Then rescale M_n by $(p_n - 1)/p_n$ (for a large prime p_n) such that $\mathcal{A}(M_n) \notin M_{n-1}$ for any $n \geq 2$.
4. Finally, take M to be the smallest Puiseux monoid containing M_n for each $n \in \mathbb{N}$. $\square$

A consequence of our First Main Result

We can use our first main result to somehow generalize the Geroldinger-Schmid Theorem.

Question: Given $S_1, S_2, \dots, S_n \subset \mathbb{P}_{\text{fin}}$, can we find a numerical monoid M and elements $x_1, x_2, \dots, x_n \in M$ such that $\mathsf{L}(x_i) = S_i$?

Corollary (First Main Result)

For all $S_1, S_2, \dots, S_n \subset \mathbb{P}_{fin}$, there exist a numerical monoid M and $x_1, x_2, \dots, x_n \in M$ such that $\mathsf{L}(x_i) = S_i$.

The Characterization Problem

Characterization Problem: Given a family $\mathcal{F}$ of atomic monoids, does $\mathcal{L}(M) = \mathcal{L}(M')$ for $M, M' \in \mathcal{F}$ always imply that $M \cong M'$?

If $\mathcal{F}$ is some family of Krull monoid, we have the next conjecture.

Conjecture (The Characterization Problem for Krull monoids)

Let M and M' be Krull monoids with respective finite abelian class groups G and G' each of their classes contains at least one prime divisor. Assume also that the Davenport constant $\mathsf{D}(G) \geq 4$. If $\mathcal{L}(M) = \mathcal{L}(M')$, then $M \cong M'$.

If $\mathcal{F}$ is the family of numerical monoid, we have a negative answer.

Theorem (Amos-Chapman-Hine-Paixao, 2007)

There are distinct (and so non-isomorphic) numerical monoids with the same system of sets of lengths.

The Characterization Problem for Pusieux monoids

Lemma 1: The homomorphisms of Puiseux monoids are precisely those given by rational multiplication.

If $\mathcal{F}$ consists of all non-finitely generated atomic Puiseux monoids, we still have a negative answer to the Characterization Problem.

Corollary (First Main Result)

There exist two non-isomorphic non-finitely generated atomic Puiseux monoids with the same system of sets of lengths.

Sketch of proof:

1. Take P_1 and P_2 to be two disjoint infinite sets of primes.
2. Construct a Puiseux monoid M_1 as in the proof of First Main Result by considering only primes in P_1 when rescaling.
3. Construct a Puiseux monoid M_2 as in the proof of First Main Result by considering only primes in P_2 when rescaling.
4. Finally, use Lemma 1 above to show that $M_1 \not\cong M_2$. $\square$

Second Main Result

Corollary (First Main Result)

There is an atomic Puiseux monoid M such that $\mathbb{P}_{fin} \subset \mathcal{L}(M)$.

Our second main result complements our first main result as condition $\mathbb{P}_{\text{fin}} \subset \mathcal{L}(M)$ is replace by $\mathbb{P}_{\text{fin}} \cap \mathcal{L}(M) = \emptyset$.

Theorem (G., 2017)

[Second Main Result] There is an atomic Puiseux monoid M such that $\mathbb{P}_{fin} \cap \mathcal{L}(M) = \emptyset$, which implies that

$$\{|\mathsf{L}(x)| : x \in M\} = \{1, \infty\}.$$

Bifurcus and anti-bifurcus Puiseux monoids

Definition: An atomic monoid M is *bifurcus* if $2 \in \mathsf{L}(x)$ for every $x \in M^\bullet \setminus \mathcal{A}(M)$.

Theorem (G.-O'Neill, 2017)

There exists a bifurcus Puiseux monoid.

Question: Do anti-bifurcus Puiseux monoids exist?

Definition: An atomic monoid M is *anti-bifurcus* if $2 \notin \mathsf{L}(x)$ for every $x \in M$ such that $|\mathsf{L}(x)| < \infty$.

Corollary (Second Main Theorem)

There exists an anti-bifurcus Puiseux monoid.

Intersection of systems of sets of lengths of numerical monoids

Theorem (Geroldinger-Schmid, 2017)

We have

$$\bigcap \mathcal{L}(M) = \big\{\{0\},\{1\},\{2\}\big\},$$

where the intersection is taken over all numerical monoids $M \subset \mathbb{N}_0$*. More precisely, for every* $s \in \mathbb{Z}_{\geq 6}$*, we have*

$$\bigcap_{|\mathcal{A}(M)|=s} \mathcal{L}(M) = \big\{\{0\},\{1\},\{2\}\big\},$$

and, for every $s \in \{2,3,4,5\}$*, we have*

$$\bigcap_{|\mathcal{A}(M)|=s} \mathcal{L}(M) = \big\{\{0\},\{1\},\{2\},\{3\}\big\}.$$

Intersection of systems of sets of lengths of Puiseux monoids

Corollary (Second Main Result)

We have

$$\bigcap \mathcal{L}(M) = \big\{\{0\}, \{1\}\big\},$$

where the intersection is taking over all nontrivial atomic Puiseux monoids.

Relation to Goldbach's conjecture

Definition: A *Goldbach's number* is a positive even integer that can be expressed as the sum of two odd primes. Let G denote the set of Goldbach's numbers.

Conjecture (Goldbach, 1742)

$\mathrm{G} = \{2n \mid n \in \mathbb{N}_{\geq 2}\}$.

Theorem (Helfgott, 2013)

Every odd $n \geq 7$ can be written as the sum of three prime numbers.

A ‘simple’ set of lengths and the Goldbach’s conjecture

Definition (elementary primary Puiseux monoid)

We call the monoid $E = \langle 1/p \mid p \text{ is prime} \rangle$ the *elementary primary Puiseux monoid*.

- $2 \in E$;
- E is (hereditarily) atomic;
- E is not a BF-monoid.

Proposition (G., 2017)

$\mathsf{L}_E(2) = \mathsf{G}$.

References

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End of Presentation

THANK YOU!