Cluster algebras: Factoriality & class groups

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Cluster algebras were 'discovered' in June 2000 by Fomin-Zelevinsky.

Google 'Cluster algebras portal' by S. Fomin.

Conference 'Cluster algebras: 20 years on' (CIRM-France)

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1011 preprints on arxiv. Mostly in interplay with other areas of mathematics (Teichmuller theory, representation theory, combinatorics, knot-theory, Lie algebras, ...), but not so many about ring theory properties.

What is a cluster algebra?

Let $n > 0$ and $m \geq 0$; and K is $\mathbb{Z}$ or a field of char 0.

$$A(\Sigma) \subset K(x_1^{\pm 1}, \ldots, x_n^{\pm 1}, \ldots, x_{n+m}^{\pm 1}) = \mathcal{F}$$

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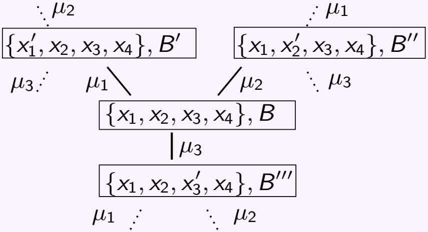
Start with a *seed* $\Sigma = (\{x_1, \ldots, x_n, \ldots, x_{n+m}\}, B)$ and find new seeds.

$\{x_1, \ldots, x_n, \ldots, x_{n+m}\}$: *cluster* B: exchange matrix   *mutation* μ_i, $i \in [1, n]$.

Example: $n = 3, m = 1$.



$B = (b_{ij}) \in M_{n+m,n}(\mathbb{Z})$, and the upper $n \times n$ part is skew-symmetrizable ($\exists d_i > 0$ such that $d_i b_{ij} = -d_j b_{ji}$).

The elements x_i ($i \in [1, n]$) are called exchangeable variables.

Mutation process

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Apply μ_i:

1. Ex. var. $x_i . x_i' = \prod_{\substack{j \in [1,n+m] \\ b_{ji} > 0}} x_j^{b_{ji}} + \prod_{\substack{j \in [1,n+m] \\ b_{ji} < 0}} x_j^{-b_{ji}} = f_i \to$ *exchange polinomyal*
2. Matrices $B' = (b'_{kl}) = \mu_i(B)$ is

$$b'_{kl} = \begin{cases} -b_{kl} & \text{if } k = i \text{ or } l = i; \\ b_{kl} + \frac{1}{2}(|b_{ki}|b_{il} + b_{ki}|b_{il}|) & \text{otherwise.} \end{cases}$$

Definition

The *cluster algebra* $A(\Sigma)$ is the K-algebra

$$A(\Sigma) = K[x, y \mid x \in \mathcal{X},\ y \in \{x_{n+1}^{\pm 1}, \ldots, x_{n+m}^{\pm 1}\}] \subseteq \mathcal{F},$$

where $\mathcal{X}$ denotes the set of exchangeable variables obtained form the seed Σ via mutations.

Two mayor theorems about cluster algebras are:

Laurent Phenomena [FZ]

Let $u \in \mathcal{X}$ be a cluster variable, then $u = \dfrac{f(x_1, \ldots, x_{n+m})}{x_1^{d_1} \ldots x_{n+m}^{d_{n+m}}}$

Positivity [LS]

For a cluster variable $u = \dfrac{f(x_1, \ldots, x_{n+m})}{x_1^{d_1} \ldots x_{n+m}^{d_{n+m}}}$, the coefficients in $f(x_1, \ldots, x_{n+m})$ are non-negative integers.

Example: For $n = 3, m = 1$, take the initial seed $((x_1, \ldots, x_4), B)$, $B = \begin{bmatrix} 0 & 0 & -2 \\ 0 & 0 & 3 \\ 2 & -3 & 0 \\ -1 & 0 & 0 \end{bmatrix}$

Some exchangeable variables are: first mutation
$(x_1^2 + x_2^3)/x_3$

In a second mutation,
$(x_1^6 + 3x_1^4x_2^3 + 3x_1^2x_2^6 + x_2^9 + x_1^6x_3^3)/(x_2x_3^3)$

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Third:
$(x_1^{36} + 18x_1^{34}x_2^3 + 153x_1^{32}x_2^6 + 816x_1^{30}x_2^9 + 3060x_1^{28}x_2^{12} + 8568x_1^{26}x_2^{15} + 18564x_1^{24}x_2^{18} + 31824x_1^{22}x_2^{21} + 43758x_1^{20}x_2^{24} + 48620x_1^{18}x_2^{27} + 43758x_1^{16}x_2^{30} + 31824x_1^{14}x_2^{33} + 18564x_1^{12}x_2^{36} + 8568x_1^{10}x_2^{39} + 3060x_1^8x_2^{42} + 816x_1^6x_2^{45} + 153x_1^4x_2^{48} + 18x_1^2x_2^{51} + x_2^{54} + (6x_1^{36} + 90x_1^{34}x_2^3 + 630x_1^{32}x_2^6 + 2730x_1^{30}x_2^9 + 8190x_1^{28}x_2^{12} + 18018x_1^{26}x_2^{15} + 30030x_1^{24}x_2^{18} + 38610x_1^{22}x_2^{21} + 38610x_1^{20}x_2^{24} + 30030x1^{18}x_2^{27} + 18018x_1^{16}x_2^{30} + 8190x_1^{14}x_2^{33} + 2730x_1^{12}x_2^{36} + 630x_1^{10}x_2^{39} + 90x_1^8x2^{42} + 6x1^6x2^{45})x_3^3 + (15x1^{36} + 180x_1^{34}x_2^3 + 990x_1^{32}x_2^6 + 3300x1^{30}x_2^9 + 7425x_1^{28}x_2^{12} + 11880x_1^{26}x_2^{15} + 13860x_1^{24}x_2^{18} + 11880x_1^{22}x_2^{21} + 7425x_1^{20}x_2^{24} + 3300x_1^{18}x_2^{27} + 990x_1^{16}x_2^{30} + 180x_1^{14}x_2^{33} + 15x_1^{12}x_2^{36})x_3^6 + (20x_1^{36} + 180x_1^{34}x_2^3 + 720x_1^{32}x_2^6 + 1680x_1^{30}x_2^9 + 2520x_1^{28}x_2^{12} + 2520x_1^{26}x_2^{15} + 1680x_1^{24}x_2^{18} + 720x_1^{22}x_2^{21} + 180x_1^{20}x_2^{24} + 20x_1^{18}x_2^{27})x_3^9 + \ldots)/(x_1x_2^6x_3^{18})$

$\Gamma(B)$ - acyclic cluster algebras - principal coefficients

One associates an oriented graph $\Gamma(B)$ to B, with vertices $[1, n+m]$.

1. for $i, j \in [1, n]$, if $b_{ij} > 0$ there are b_{ij} arrows $i \to j$.
2. for $i \in [n+1, m]$ if $b_{ij>0}$ there are b_{ij} arrows $i \to j$; if $b_{ij} < 0$ there are b_{ij} arrows $j \to i$.

Definition

The seed Σ is *acyclic* if the full subgraph of $\Gamma(B)$ on the vertices $[1, n]$ is acyclic.

Definition

$A(\Sigma)$ has *principal coefficients* if $n = m$ and the lower $n \times n$ part of B is the identity matrix.

Some results

If Σ **is acyclic**, then:

Theorem [BFZ07]

$A(\Sigma)$ is f.g. and noetherian. There is an isomorphism

$$K\left[X_i, X_i', X_j^{\pm 1} \;\middle|\; i \in [1,n], j \in [n+1, n+m]\right] / \langle X_i X_i' - f_i \mid i \in [1,n] \rangle \;\simeq\; A(\Sigma).$$

Theorem [Mull14]

[Mul14] If $A(\Sigma)$ is locally acyclic, is integrally closed.

Krull domains

Let A be a domain and $\mathbf{q}(A)$ its field of fractions.

$x \in \mathbf{q}(A)$ is *almost integral* (over A) if there exists $c \in \mathbf{q}(A)^\times$ such that $cx^n \in A\ \forall n \geq 0$.

A is *completely integrally closed* if every almost integral element $x \in \mathbf{q}(A)$ belongs to A.

Krull domain

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Krull domain

A *Krull domain* is a domain A that is v-noetherian and completely integrally closed.

1. noetherian $\Rightarrow$ v-noetherian.
2. A noetherian domain is completely integrally closed if and only if it is integrally closed.

From previous slide: there are cluster algebras (for ex. acyclic) that are Krull domains.

Class group of a Krull domain A

A *fractional ideal* is an A-submodule $\mathfrak{a} \subseteq \mathbf{q}(A)$ such that there exists an $x \in \mathbf{q}(A)^\times$ with $x\mathfrak{a} \subseteq A$.

For a fractional ideal $\mathfrak{a}$, let

$$\mathfrak{a}^{-1} = (A : \mathfrak{a}) = \{\, x \in \mathbf{q}(A) \mid x\mathfrak{a} \subseteq A \,\} \quad \text{and} \quad \mathfrak{a}_v = (\mathfrak{a}^{-1})^{-1}.$$

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Take

1. $\mathcal{F}_v(A)^\times$ the set of nonzero divisorial fractional ideals.
2. $\mathcal{H}(A) = \{\, xA \mid x \in \mathbf{q}(A)^\times \,\} \subseteq \mathcal{F}_v(A)^\times$ the subgroup of nonzero principal fractional ideals.

Definition

The (*divisor*) *class group* of A is $\mathcal{C}(A) = \mathcal{F}_v(A)^\times / \mathcal{H}(A)$.

Theorem

Let A be a domain, TFAE:

1. A is factorial.
2. A is atomic and every atom of A is a prime element.
3. A is a Krull domain and $\mathcal{C}(A)$ is trivial.

For cluster algebras, Geiss–Leclerc–Schröer have shown the following.

Theorem [GLS]

Let $\Sigma = (\mathbf{x}, B)$ be a seed, $\mathbf{x} = (x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m})$. Let $A = A(\Sigma)$.

1. Any exch. cluster variable is an atom.
2. The group of units of A is

$$A^\times = K^\times \times \langle x_j^{\pm 1} \mid j \in [n+1, n+m] \rangle.$$

First main result

Theorem [-LS]

Let $A(\Sigma)$ be a cluster algebra that is a Krull domain. Let $t \in \mathbb{Z}_{\geq 0}$ denote the number of height-1 prime ideals that contain one of the exchangeable variables $x_1, \dots, x_n$.

1. The class group $\mathcal{C}(A(\Sigma))$ is a free abelian group of rank $t - n$.
2. If $n + m > 0$, that is $A(\Sigma) \neq K$, then each class contains exactly $\mathrm{card}(K)$ height-1 prime ideals.

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A result from Keinrath, implies that:

Corollary

Let $A(\Sigma)$ be a cluster algebra that is a Krull domain and suppose that $A(\Sigma)$ is not factorial. Let $L \subseteq \mathbb{Z}_{\geq 2}$ be a finite set. Then there exists an element $a \in A(\Sigma)^{\bullet}$ such that $\mathsf{L}(a) = L$.

Acyclic case

Let $A(\Sigma)$ be an acyclic cluster algebra. We define an equivalence relation, and equivalence classes over the set of vertices $[1, n]$ of $\Gamma(B)$.

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Partners

1. Two indices i, $j \in [1, n]$ are *partners* if the exchange polynomials f_i and f_j have a non-trivial common factor in $K[\mathbf{x}]$.
2. Partnership is an equivalence relation on the set $[1, n]$. An equivalence class is called a *partner set*.

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Partnership is very easy to compute from $\Gamma(B)$.

The factorization of the exchange polynomials f_i is in correspondence with the factorization of the cyclotomic polynomial Φ_d over K.

Theorem [-LS]

Let A be a cluster algebra with acyclic seed Σ. Then TFAE.

1. A is factorial.
2. The exchange polynomials $f_1, \ldots, f_n$ are prime elements in $K[\mathbf{x}]$ and pairwise distinct.

Second main result

Theorem [-LS]

Let A be a cluster algebra with acyclic seed Σ. Then TFAE.

1. A is factorial.
2. The exchange polynomials $f_1, \ldots, f_n$ are prime elements in $K[\mathbf{x}]$ and pairwise distinct.

Corollary

Let B be skew-symmetric and s.t. $\Gamma(B)$ has no parallel arrows and $\Sigma = (\mathbf{x}, B)$. Then $A(\Sigma)$ is factorial if and only if every partner set V is a singleton. (That is, $\Gamma(B)$ admits no partners $i \neq j$).

Corollary

Suppose that $\Sigma = (\mathbf{x}, B)$ is acyclic and has principal coefficients. Then the cluster algebra $A(\Sigma)$ is factorial.

Third main result

$\mu_d^*(K)$: set of d-th primitive roots of unity in K; d_i: g.c.d (column i of B)

$\nu_K(d)$: the number of irreducible factors of the cyclotomic polynomial Φ_d over K.

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Theorem [-LS]

Let $\Sigma = (\mathbf{x}, B)$ be an acyclic seed. For a partner set $V \subseteq [1, n]$ and $d \in \mathbb{Z}_{\geq 1}$, let

- $\mathrm{c}(V, d)$ denote the number of $i \in V$ for which d divides d_i,
- $\mathrm{e}(V) = \mathrm{v}_2(\mathrm{d}_i)$ be the 2-valuation of d_i for $i \in V$ (this is independent of i).

Then the class group of $A(\Sigma)$ is a finitely generated free abelian group of rank

$$r = \sum_{\substack{V \\ V \text{ a partner set}}} r_V,$$

where

$$r_V = 2^{|V|} - 1 - |V| \quad \text{if } V \text{ is the partner set of isolated indices,}$$

and otherwise

$$r_V = \sum_{\substack{d \in \mathbb{Z}_{\geq 1} \\ d \text{ odd}}} \left(2^{\mathrm{c}(V,d)} - 1\right) \nu_K(2^{\mathrm{e}(V)+1} d) - |V|.$$

Example

Recall the example (long ago), $n = 3, m = 1$, take the initial seed $\Sigma = ((x_1, \dots, x_4), B)$,

$$B = \begin{bmatrix} 0 & 0 & -2 \\ 0 & 0 & 3 \\ 2 & -3 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

The exchange polynomials are

$$f_1 = x_3^2 + x_4 \;\; ; \;\; f_2 = x_3^3 + 1 = (x_3 + 1)(x_3^2 - x_3 + 1) \;\; ; \;\; f_3 = x_1^2 + x_3^3$$

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Also $\mathrm{d}_1 = 1, \mathrm{d}_2 = 3, \mathrm{d}_3 = 1$. There are no common factors, so the partner sets are singletons. If $K = \mathbb{Z}$ (or $\mathbb{Q}$) then $\nu_K(2^{\mathrm{e}(V)+1}d) = 1$ for all d.

$$r = \sum_V r_V = 2.[2^{\mathrm{c}(\{1\},1)} - 1 - |\{1\}|] + [2^{\mathrm{c}(\{2\},1)} - 1 + 2^{\mathrm{c}(\{2\},3)} - 1 - |\{2\}|] = 1$$

For $A(\Sigma)$, considered as a $\mathbb{Q}$-algebra, the class group is $\mathbb{Z}$.

Further questions

- Explore the case of locally acyclic cluster algebras. 'Cluster like' algebras, like LP algebras.
- Every locally acyclic cluster algebra is a Krull domain. Not every cluster algebra is a Krull domain. We lack an exact classification of which cluster algebras are Krull domains.
- Investigate the divisor-closed submonoid of a cluster algebra generated by its initial cluster, respectively, by all cluster variables.
- Any Krull domain A possesses a *transfer homomorphism* to a *monoid of zero-sum sequences* $\mathcal{B}(G_0)$, where G_0 is the subset of the class group of A containing height-1 prime ideals. The atoms in $\mathcal{B}(G_0)$ are the minimal zero-sum sequences over G_0. If A is a cluster algebra, each cluster variable is an atom, and hence gives rise to such a minimal zero-sum sequence. It may be interesting to see which minimal zero-sum sequences arise in this way.
- Cluster algebra machinery is applied in a lot of contexts in mathematics. The fact that a certain cluster algebra is factorial may be used in these contexts.

For more on this ...

Google

Factoriality and class groups of cluster algebras.

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Thank you for your attention. I hope you had a good time in Graz.