New results on the Noether bound

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Conference on Rings and Factorisations, Graz

20 February, 2018

Definition of the Noether number

- Let $G \subset \mathrm{GL}(V)$ be a finite group acting on an n-dimensional vector space V over a field $\mathbb{F}$,
- The action of G extends to the polynomial ring $\mathbb{F}[x_1, ..., x_n]$ where the variables $x_1, ..., x_n$ form a basis of the dual V^*:
$$(g \cdot f)(x_1, ..., x_n) = f(g^{-1}(x_1, ..., x_n)) \qquad g \in G, f \in \mathbb{F}[V]$$
- The ring of polynomial invariants of the G-module V is
$$\mathbb{F}[V]^G := \{f \in \mathbb{F}[V] : g \cdot f = f \text{ for all } g \in G\}$$
- $\beta(G, V) = \min\{s \in \mathbb{N} : \mathbb{F}[V]^G \text{ is generated by } \bigoplus_{d=0}^{s} \mathbb{F}[V]_d^G\}$
$\beta(G) = \beta_{\mathbb{F}}(G) = \sup\{\beta(G, V) : V \text{ is a } G\text{-module over } \mathbb{F}\}$

The case of Abelian groups

If G is abelian then all irreducible G-modules are 1-dimensional.
$\Rightarrow$ In a suitable basis each element of A acts by diagonal matrices.
$\Rightarrow$ The variables in $\mathbb{C}[V] = \mathbb{C}[x_1, ..., x_n]$ can be chosen so that :

$$a \cdot x_i = \theta_i(a) x_i \quad \text{for each } a \in A$$

The character $\theta_i \in \mathrm{Hom}(A, \mathbb{C}^\times) \cong A$ is called the *weight* of x_i.

monomial $x_1^{e_1} \cdots x_n^{e_n}$	$\leftrightarrow$	sequence of the weights
monomials spanning $\mathbb{C}[V]^A$	$\leftrightarrow$	zero-sum sequences over A
generators of $\mathbb{C}[V]^A$	$\leftrightarrow$	minimal zero-sum sequences
Noether number $\beta(A)$	$\leftrightarrow$	Davenport constant $\mathsf{D}(A)$

Generalising the Davenport constant for non-abelian groups

- The sequences $(g_1, g_2, \ldots, g_n) \in G$ form a free abelian monoid $\mathcal{F}(G)$ with respect to concatenation.
- A *product-one sequence* is such that $g_{\pi(1)} g_{\pi(2)} \cdots g_{\pi(n)} = 1$ for some $\pi \in S_n$. They form a submonoid $\mathcal{B}(G) \subset \mathcal{F}(G)$.
- $\mathsf{D}(G)$ denotes the maximal length of an atom (i.e. irreducible element) of $\mathcal{B}(G)$
- $\mathsf{d}(G)$ denotes the maximal length of a product-one free sequence, i.e. an element of $\mathcal{F}(G)$ not divisible by any element of $\mathcal{B}(G)$.

Previous results on the Davenport constant

- $\mathsf{d}(G) + 1 \leq \mathsf{D}(G) \leq |G|$ is easily seen.
- (Olson–White, 1977) For any non-cyclic group G we have $\mathsf{d}(G) \leq \frac{1}{2}|G|$
- Equality holds only if G contains a cyclic subgroup of index 2 and in this case $\mathsf{D}(G) = \mathsf{d}(G) + |G'|$ (Geroldinger–Grynkiewicz, 2013)
- (Grynkiewicz 2013) For $C_p \rtimes C_q$ where $q \mid p - 1$ we have $\mathsf{d}(G) = p + q - 2$ and $\mathsf{D}(G) = 2p$.

Previous results on the Noether number

- $\beta(G) \leq |G|$ (Noether 1916, Fleischman–Fogarty 2000) and this upper bound is sharp only if G is cyclic (Schmid 1989)
- (Cz. – Domokos, 2013) For any non-cyclic group G we have $\beta(G) < \frac{1}{2}|G|$ unless G contains a cyclic subgroup of index 2 or it is one of four particular groups of small order.
- If G is non-cyclic with a cyclic subgroup of index 2 then $\beta(G) = \frac{1}{2}|G| + \begin{cases} 2 & \text{if } G \text{ is dicyclic} \\ 1 & \text{otherwise} \end{cases}.$
- (Pawale's conjecture) For two primes p, q such that $q \mid p-1$ the semidirect product $G = C_p \rtimes C_q$ has $\beta(G) = p + q - 1$. (Proven for $q = 2, 3$.)

Is there a connection between $\mathsf{d}(G)$, $\mathsf{D}(G)$ and $\beta(G)$?

For any abelian group A it is easily seen from the correspondence between zero-sum sequences and A-invariant monomials that

$$\mathsf{d}(A) + 1 = \beta(A) = \mathsf{D}(A).$$

In all the previously mentioned cases we have also observed that

$$\mathsf{d}(G) + 1 \leq \beta(G) \leq \mathsf{D}(G).$$

Can this be in general the case? If so, is there any structural connection which could account for this phenomenon?

This question lead us to calculate the Noether number and the Davenport constants for all groups of order less than 32.

Summary of our results

GAP	G	d	β	D
$(12,3)$	A_4	4	6	7
$(16,11)$	$Dih_8 \times C_2 \ = \ (C_4 \times C_2) \rtimes_{-1} C_2$	5	6	7
$(16,3)$	$K_4 \rtimes C_4 = (C_4 \times C_2) \rtimes_{\psi} C_2$	5	6	7
$(16,13)$	$(Pauli) \ = \ (C_4 \times C_2) \rtimes_{\phi} C_2$	5	7	7
$(16,12)$	$Q_8 \times C_2$	5	7	7
$(16,4)$	$C_4 \rtimes C_4$	6	7	8
$(18,3)$	$S_3 \times C_3$	7	8	10
$(18,4)$	$(C_3 \times C_3) \rtimes_{-1} C_2$	5	6	10
$(20,3)$	$C_5 \rtimes C_4$	7	8	10
$(21,1)$	$C_7 \rtimes C_3$	8	9	14
$(24,8)$	$C_3 \rtimes Dih_8 = (C_6 \times C_2) \rtimes_{\gamma} C_2$	7	9	14
$(24,12)$	S_4	6	9	12
$(24,13)$	$A_4 \times C_2$	7	8	10
$(24,7)$	$Dic_{12} \times C_2$	8	9	11
$(24,14)$	$Dih_{12} \times C_2$	7	8	10
$(24,3)$	$SL(2,3) = \tilde{A}_4$	7	12	13
$(27,4)$	$M_{27} = C_9 \rtimes C_3$	10	11	12

An interesting case: the Heisenberg group

For any prime $p > 2$ the Heisenberg group is the group of order p^3 $H_p = \langle a, b \rangle$ subject to the relations

$$a^p = b^p = c^p = 1 \quad c = [a, b] \quad [a, c] = [b, c] = 1.$$

Then the center and the derived subgroup of H_p coincide with $\langle c \rangle$ (it is an extraspecial group) and $H_p / \langle c \rangle \cong C_p \times C_p$.

Theorem (Cz. 2015)

Let $p \geq 5$ and assume that the characteristic of $\mathbb{F}$ is larger than p. Then we have

$$\beta_{\mathbb{F}}(H_p) < p^2.$$

However for $p = 3$ and $\mathrm{char}(\mathbb{F}) \neq 3$ we have $\beta_{\mathbb{F}}(H_{27}) = 9$.

Lower bound for the case $p = 3$?

H_{27} is acting on $\mathbb{C}[x, y, z]$ as follows ($\zeta = e^{2\pi i/3}$):

$$a = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & \zeta^2 \end{bmatrix}, \quad b = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad c = \zeta \cdot Id$$

Any element of $R := \mathbb{C}[x, y, z]^{H_{27}}$ must have degree divisible by 3. $R_3 \oplus R_6$ is spanned by xyz, $x^3 + y^3 + z^3$ and $x^3y^3 + y^3z^3 + z^3x^3$. These are all symmetric polynomials so if we had $\beta(H_{27}) < 9$ then necessarily $R \subset \mathbb{C}[x, y, z]^{S_3}$. But there exists a H_{27}-invariant polynomial which is not symmetric, namely $x^6y^3 + y^6z^3 + z^6x^3$.

Algorithms to calculate the Davenport constants

To make the calculations feasible we had to deal with the overwhelming combinatorial complexity. We used some tricks:

- We enumerated atoms by splitting smaller atoms, i.e. replacing an element g in the sequence by pairs $(x, x^{-1}g)$
- We enumerated product-one sequences and atoms only up to similarity, i.e. the action of the group $\mathrm{Aut}(G)$

  E.g. H_{27} has 108827 atoms grouped in only 340 similarity classes. This way we have found that

$$\mathsf{D}(H_{27}) = 8.$$

This is a counterexample to the conjecture $\beta(G) \leq \mathsf{D}(G)$!

A byproduct of our inquiry

- It was known a long time ago that for any subgroup $H \leq G$ we have $\beta(H) \leq \beta(G)$.
- Is it possible that $\beta(G) = \beta(H)$ for a proper subgroup $H < G$?
- We have found no such example on our list. Since then we have proved that this is not possible:

Theorem (Cz – Domokos, 2017)

For every proper subgroup $H < G$ we have $\beta(H) < \beta(G)$. Even more, for an arbitrary normal subgroup $N \triangleleft G$ we have

$$\beta(G) \geq \beta(N) + \beta(G/N) - 1.$$

Thank you for your attention!