Pointwise minimal extensions

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Graz, February 20, 2018

What is it about?

Pointwise minimal extensions were introduced by P.-J. Cahen, D. E. Dobbs and T. G. Lucas in the context of domains [Valuative domains, *J. Algebra Appl.*, (2010)].

Definition

A ring extension $R \subset S$ is said to be *minimal* if there is no ring properly between R and S.

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Notations, hypotheses, vocabulary

All rings are commutative with 1.

"$\subset$" denotes proper containment.

$R \subset S$ is a ring extension (same 1).

An element $t \in S$ is said to be *minimal* (over R) if $R \subset R[t]$ is a minimal extension

$t \in S$ is said to be *trivial* if $t \in R$. Thus (alternate definition):

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Minimal extensions were introduced by Ferrand and Olivier: Homomorphismes minimaux d'anneaux, *J. Algebra*, (1970)

They have often been considered, also in relation with chains of minimal extensions.

Obviously, if $R \subset S$ is a minimal extension, either S is integral over R or R is integrally closed in S.

And obviously also, S is integral over R if and only if it is *finite* (finitely generated as an R-module).

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Ferrand and Olivier's first result

Theorem (Ferrand Olivier)

Let $R \subset S$ be a minimal extension.

1. *There exist a maximal ideal M of R (the crucial ideal of the extension) such that, for each prime $P \neq M$, $R_P = S_P$.*
2. *Either $R \subset S$ is finite and $MS = M$, or $R \subset S$ is closed and $MS = S$.*

And for pointwise minimal extensions...

The dichotomy integral/integrally closed extends (almost) perfectly to pointwise minimal extensions (already seen by Cahen, Dobbs, Lucas in the case of domains).

However, an integral pointwise minimal extension need not be finite.

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The closed case

The closed case turns out to be trivial.

Proposition

An integrally closed extension $R \subset S$ is pointwise minimal if and only if it is minimal.

Example

$V \subset K$, with V a rank-one valuation domain. K its quotient field.

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The integral case

For an integral extension. The rings R and S share the ideal M.

Corollary

An integral extension $R \subset S$ is minimal (resp. pointwise minimal) if and only if there is a maximal ideal M of R such that $MS = M$ and $R/M \subset S/M$ is minimal (resp. pointwise minimal).

Proof: The rings between R/M and S/M are in one-one order preserving correspondence with the rings between R and S.

We can then reduce to extensions over a field.

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Three type of minimal extensions over a field

Lemma (Ferrand and Olivier)

Let k be a field and $f : k \to A$ be an injective morphism. Then f is minimal if and only if of one of the three following types:

1. *A is a field and $k \to A$ is minimal (inert).*
2. *f is the diagonal morphism $k \to k \times k$ (decomposed).*
3. *f is the canonical morphism $k \to D_k(k) := k[X]/(X^2)$ (ramified).*

In particular A is a finite k-algebra.

Similarly (but changing the order).

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Minimal integral extensions

Proposition

Let $R \subset S$ be a minimal integral extension with crucial ideal M. There are three cases:

1. $S/M \cong (R/M)^2$ *(decomposed),*
2. $R/M \subset S/M$ *is a minimal field extension (inert),*
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Definition

Let $R \subset S$ be an extension. We say that $x \in S$ is *decomposed* (resp. inert, ramified) if $R \subset R[x]$ is minimal decomposed (resp. inert, ramified).

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Integral extensions over a field

In this section, we let $k \subset S$ be an extension over a field.

- Decomposed and ramified elements are of degree 2,
- a non trivial idempotent e is decomposed,
- a nilpotent element x is ramified if and only if of degree 2, that is $x \neq 0$, and $x^2 = 0$,
- an inert element can be of any degree.

Proposition

If $k \subset S$ is integral, every finite sub-extension, in particular every simple sub-extension $k \subset k[t]$, contains a minimal sub-extension.

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Decomposed with other ones

Lemma

1. *If $e \in S$ is a non trivial idempotent and $x \in S$ a nilpotent element of degree 2, then $(e + x)$ is neither trivial nor minimal.*
2. *If $e \in S$ is a non trivial idempotent and $y \in S$ is inert, then ey is neither trivial nor minimal.*

Thus, if $k \subset S$ is pointwise minimal, decomposed elements don't mix with other ones.In general, all three types may coexist:

Example

$k = \mathbb{R}$ and $S := \mathbb{C} \times \mathbb{C}[X]/(X^2)$.
(i, i) is inert, $(1, 0)$ is a non trivial idempotent, $(0, x)$ (with x the class of X) is a non-zero nilpotent element of degree 2.

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Ramified with inert

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Example

The extension $\mathbb{R} \subset S = \mathbb{C}[X]/(X^2)$ is finite: $\dim_{\mathbb{R}}(S) = 4$.

$i \in \mathbb{C}$ is inert, and the class x of X is nilpotent of degree 2.

As $\mathrm{char}(\mathbb{R}) = 0$, it follows that $(i + x)$ is not minimal.

In fact, $\mathbb{R}[i + x] = S$.

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More on decomposed and ramified elements

We now always suppose that $k \subset S$ is an integral extension.

Lemma

There is a decomposed element if and only if S is not a local ring.

Lemma

Let N be the nilradical (and also Jacobson radical of S).
A minimal element $x \in S$ is ramified if and only if $x \in k + N$ (otherwise inert or decomposed).
A non trivial $x \in N$ is ramified if and only if $x^2 = 0$.

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The four types of pointwise minimal extensions

As decomposed elements do not mix with other ones, four cases may occur:

Definition

We say a pointwise minimal (integral) extension is *decomposed* (resp. *inert*, resp. *ramified*), if all minimal sub-extensions are decomposed (resp. inert, resp. ramified). We say it is *composite* if there are both ramified and inert sub-extensions.

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Proposition

Let $k \subset S$ be a pointwise minimal extension.

1. *If S is not a local ring, then $k \subset S$ is decomposed.*
2. *If S is a local ring with maximal ideal N, then all minimal sub-extensions are either ramified or inert. More precisely,*
   - *the extension is inert if and only if S is a field, equivalently $N = (0)$,*
   - *ramified if and only if $S = k + N$,*
   - *composite if and only if $k \subset k + N \subset S$.*

We shall see that all four cases may effectively occur.

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We shall see that all four cases may effectively occur.

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General integral extensions

We now consider an integral extension $R \subset S$. We derive the following:

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Let $R \subset S$ be a pointwise minimal integral extension with crucial ideal M (the maximal ideal such that $MS = M$).

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Theorem

An integral extension $R \subset S$ is pointwise minimal if and only if there is a maximal ideal M of R such that $MS = M$ and, letting $k = R/M$, one of the following (mutually exclusive) conditions is satisfied.

1. *Decomposed: either $S/M \cong k^2$ or $k = \mathbb{F}_2$ and S/M is Boolean. (Every finite sub extension is isomorphic to $\mathbb{F}_2^n$).*
2. *Inert: S/M is a field and either $k \subset S/M$ is a separable minimal field extension or $\mathrm{char}(k) = p$ and for all $x \in S, x^p \in R$. (M is maximal in S, and of course the only maximal ideal above M).*

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Characterization (continued)

Theorem (Continued)

1. *Ramified: there is an ideal I of S such that $S = R + I$, and for all $x \in I, x^2 \in M$. Equivalently, there is only one maximal ideal N of S above M, $S = R + N$, and for all $x \in N, x^2 \in M$.*
2. *Composite: there is only one maximal ideal N of S above M, $R \subset R + N \subset S$, for all $x \in N, x^2 \in M$, $\mathrm{char}(k) = p$ and for all $x \in S, x^p \in R$.*

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Examples

Examples are given by a pullback.

$$\begin{array}{ccc} R & \longrightarrow & R/M \cong k \\ \downarrow & & \downarrow \\ S & \longrightarrow & S/M \cong S' \end{array}$$

Start with a pointwise minimal extension $k \subset S'$ over a field. Take S to be a ring with an ideal M such that $S/M \cong S'$. Finally let $R = \varphi^{-1}(R/M)$, where φ is the canonical map $S \to S/M$.

Example

Given $k \subset S'$, let $S = S'[X_1, \ldots, X_n]$. Take $M = (X_1, \ldots, X_n)$, and $R = k + M$.

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1. (Minimal extension): $k \subset k^2$, for any field k.
2. $k = \mathbb{F}_2$, X any set: $\mathbb{F}_2 \subset \mathbb{F}_2^X$.

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In fact pointwise minimal field extensions (already in the paper with D.E. Dobbs and T.G. Lucas).

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1. (Minimal extension): $\mathbb{R} \subset \mathbb{C}$, or any minimal field extension.
2. $\mathrm{char}(k) = p$, I any set: $k(X_i)_{i \in I} \subset k(T_i)_{i \in I}$, where X_i, T_i are indeterminates such that $T_i^p = X_i$.

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   $k \subset S =: k[\{X_i\}_{i\in I}]/(\{X_i^2, X_iX_j\}_{i,j\in I})$.
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In both cases S is a local ring with maximal ideal $N = (X_i)_{i\in I}$.
In the first case $N^2 = (0)$.
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Composite (alternate characterization)

Proposition

$k \subset L$ is a pointwise minimal field extension with $c(k) = p$ and for all $x \in L, x^p \in k$. (As above.)
$S = L + N$, where N is a non-zero ideal of S such that, for all $x \in N, x^2 = 0$. (As above, case 1 if $p \neq 2$, case 1 or 2 if $p = 2$.)
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We give an example of minimal length:

Example

$k = \mathbb{F}_2(Y)$, $L = \mathbb{F}_2(T)$ with $T^2 = Y$, and $S := L[X]/(X^2)$.
S is a local ring with maximal ideal $N = Lx$ (where x denotes the class of X).

$$k \subset k[x] \subset k + N \subset S$$

is a maximal chain, thus $\ell[k, S] = 3$.
On the other hand,

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The end

Thank you for your attention.