Nonnoetherian coordinate rings and their noncommutative resolutions

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Notation:
$*$ All algebras over $k = \bar{k}$.
$*$ $\operatorname{Max} S$ and $\operatorname{Spec} S$ denote the maximal and prime ideal spectra of S, or variety and scheme with global sections S.

Motivation... In string theory ($\sim$ 2008), models were studied where the extra 6 dimensions of the universe happened to be described by certain nonnoetherian rings of functions.

Physicists asked: what does this geometry look like?

Consider

$$S = k[x, y] \quad \text{and} \quad R = k[x, xy, xy^2, \ldots] = k + xS.$$

$\operatorname{Max} R$ may be viewed as 2-dimensional affine space $\mathbb{A}^2 = \operatorname{Max} S$ with the line

$$\mathcal{Z}(x) = \{x = 0\} \subset \mathbb{A}^2$$

identified as single closed point.
From this perspective, $\mathcal{Z}(x)$ is a 1-dimensional 'smeared-out' point of $\operatorname{Max} R$.

Now let S be an integral domain and f.g. k-algebra, and R subalgebra of S. Set

$$U_{S/R} := \{\mathfrak{n} \in \operatorname{Max} S \mid R_{\mathfrak{n}\cap R} = S_{\mathfrak{n}}\}.$$

Proposition

Suppose $U_{S/R} \neq \emptyset$. Then

1. *$U_{S/R}$ is open in $\operatorname{Max} S$.*
2. $\dim R = \dim S$.
3. *$\operatorname{Max} R$ and $\operatorname{Max} S$ are birationally equivalent.*

In our example, $U_{S/R} = \mathcal{Z}(x)^c$.

Definition

- R is *depicted* by S if

$$\iota_{S/R} : \operatorname{Spec} S \to \operatorname{Spec} R, \quad \mathfrak{q} \mapsto \mathfrak{q} \cap R,$$

is surjective, and

$$U_{S/R} = \{\mathfrak{n} \in \operatorname{Max} S \mid R_{\mathfrak{n} \cap R} \text{ is noetherian } \} \neq \emptyset.$$

- The *geometric height* of a point $\mathfrak{p} \in \operatorname{Spec} R$ is

$$\operatorname{ght} \mathfrak{p} := \min \left\{ \operatorname{ht}(\mathfrak{q}) \mid \mathfrak{q} \in \iota_{S/R}^{-1}(\mathfrak{p}),\ S \text{ a depiction of } R \right\}.$$

The *geometric dimension* of $\mathfrak{p}$ is

$$\operatorname{gdim} \mathfrak{p} := \dim R - \operatorname{ght} \mathfrak{p}.$$

In our example, R is depicted by S, and

$$\operatorname{ght}_R(xS) = 1, \quad \text{whereas} \quad \operatorname{ht}_S(xS) = 2.$$

Theorem

Suppose R is depicted by S. Let $\mathfrak{p} \in \operatorname{Spec} R$. Then

$$\operatorname{ght}(\mathfrak{p}) \leq \operatorname{ht}_R(\mathfrak{p}),$$

with equality if there is $\mathfrak{q} \in \operatorname{Spec} S$ such that $\mathfrak{q} \cap R = \mathfrak{p}$ and

$$\mathcal{Z}_S(\mathfrak{q}) \cap U_{S/R} \neq \emptyset.$$

Furthermore, TFAE:

1. *R is noetherian.*
2. *$U_{S/R} = \operatorname{Max} S$.*
3. *$R = S$.*

In particular, if R is noetherian, then its only depiction is itself.

Question

Given algebraic sets $Y_1, \ldots, Y_n \subset \operatorname{Max} S$, does $\exists R \subset S$ such that each Y_i is a closed point of $\operatorname{Max} R$?

Let $X = \operatorname{Max} S$,
and $Y_1, \ldots, Y_n$ be a collection of non-intersecting proper algebraic sets of X.

Theorem

Let

$$R := \cap_i (k + I(Y_i)).$$

Then $\operatorname{Max} R \cong X$ *except that each* Y_i *is identified as distinct closed point. In particular,*

$$U_{S/R} = \cap_i Y_i^c.$$

Furthermore,

- R *is nonnoetherian* $\iff$ $\exists i$ *s.t.* $\dim Y_i \geq 1$.
- R *is depicted by* S $\iff$ $\forall i$, $\dim Y_i \geq 1$.

Let $S = k[x, y]$ and consider the three lines

$$\mathcal{Z}(x) = \{x = 0\}, \quad \mathcal{Z}(x-1) = \{x = 1\}, \quad \mathcal{Z}(x-2) = \{x = 2\}.$$

Then the ring

$$R = (k+xS)\cap(k+(x-1)S)\cap(k+(x-2)S) = k[x]+x(x-1)(x-2)S$$

is nonnoetherian and depicted by S.

Corollary

Let I be a nonzero proper non-maximal radical ideal of S. Set $R = k + I$. Then TFAE:

1. $\dim(S/I) \geq 1$.
2. *R is nonnoetherian.*
3. *R is depicted by S.*

Geometric height gives a geometric picture of nonnoetherian 'coordinate rings' using depictions, but does it play any role algebraically?

Yes!

...in the noncommutative resolutions of nonnoetherian singularities

Question

Let K be the function field of an algebraic variety. A subset $\mathfrak{p}$ of K may be an ideal in different subalgebras of K, and the height of $\mathfrak{p}$ depends on the choice of such subalgebra. Is the geometric height of $\mathfrak{p}$ independent of the choice of subalgebra for which $\mathfrak{p}$ is an ideal? If this is the case, then the geometric height would be an intrinsic property of an ideal, whereas its height would not be.

Let $(R, \mathfrak{m})$ be a noetherian local ring with $R/\mathfrak{m} \cong k$.

- (1950's) Auslander, Buchsbaum, Serre:

$$R \text{ regular} \iff \operatorname{gldim} R = \operatorname{pd}_R(k) = \dim R = \operatorname{ht}(\mathfrak{m}).$$

- (1984) Brown and Hajarnavis:

A - noncommutative noetherian ring, f.g. module over its center R. A is *homologically homogeneous* (hom hom) if for each simple A-module V,

$$\operatorname{gldim} A = \operatorname{pd}_A(V) = \dim R = \operatorname{ht}(ann_R(V)).$$

- (2000) string theory...

A is a *noncommutative resolution* (NCR) if A is hom hom, and

$$A \otimes_R \operatorname{Frac} R \sim_{\text{Morita}} \operatorname{Frac} R.$$

- (2001) Van den Bergh:

A is a *noncommutative crepant resolution* (NCCR) if R is a normal Gorenstein domain, A is hom hom, and

$$A \cong \operatorname{End}_R(M),$$

with M a f.g. reflexive R-module.

Let B be an integral domain and k-algebra, and let

$$A = \left[A^{ij}\right] \subset M_d(B)$$

be a tiled matrix ring, i.e., each $A^i := A^{ii} \subset B$ is unital.

Definition

Set

$$R := k\left[\cap_{i=1}^{d} A^i\right] \quad \text{and} \quad S := k\left[\cup_{i=1}^{d} A^i\right].$$

The *cyclic localization* of A at $\mathfrak{q} \in \operatorname{Spec} S$ is

$$A_{\mathfrak{q}} := \left\langle \begin{bmatrix} A^1_{\mathfrak{q}\cap A^1} & A^{12} & \cdots & A^{1d} \\ A^{21} & A^2_{\mathfrak{q}\cap A^2} & & \\ \vdots & & \ddots & \vdots \\ A^{d1} & & \cdots & A^d_{\mathfrak{q}\cap A^d} \end{bmatrix} \right\rangle \subset M_d(\operatorname{Frac} B).$$

$*$ In cases of interest, $Z := Z(A) \cong R$ and R is depicted by S.

$*$ If $R = S$, then $A_{\mathfrak{q}} \cong A \otimes_R R_{\mathfrak{q}}$.

Suppose B is f.g. over k and k is uncountable. Further suppose

- for generic $\mathfrak{b} \in \operatorname{Max} B$, the composition

$$A \hookrightarrow M_d(B) \xrightarrow{1} M_d(B/\mathfrak{b})$$

  is surjective;

- the morphism

$$\operatorname{Max} B \to \operatorname{Max} Z, \quad \mathfrak{b} \mapsto \mathfrak{b}\mathbf{1}_d \cap Z,$$

  is surjective; and

- for each $\mathfrak{n} \in \operatorname{Max} S$, $R_{\mathfrak{n} \cap R} = S_{\mathfrak{n}}$ iff $R_{\mathfrak{n} \cap R}$ is noetherian.

Then $Z = R\mathbf{1}_d$, and R is depicted by S.
Furthermore,

$$\begin{aligned} R = S \quad &\Leftrightarrow \quad A \text{ is a finitely generated } R\text{-module} \\ &\Leftrightarrow \quad R \text{ is noetherian} \\ &\Rightarrow \quad A \text{ is noetherian} \end{aligned}$$

Definition

- A is *cycle regular* if $\forall \mathfrak{q} \in \operatorname{Spec} S$ minimal over $\mathfrak{m}$, and $\forall$ simple $A_{\mathfrak{q}}$-module V,

$$\operatorname{gldim} A_{\mathfrak{q}} = \operatorname{pd}_{A_{\mathfrak{q}}}(V) = \dim S_{\mathfrak{q}} = \operatorname{ght}\left(\operatorname{ann}_{Z(A_{\mathfrak{q}})} V\right).$$

- A is a *nonnoetherian NCR* if A is cycle regular, and

$$A \otimes_R \operatorname{Frac} R \sim_{\text{Morita}} \operatorname{Frac} R.$$

- A is a *nonnoetherian NCCR* if S is a normal Gorenstein domain, A is cycle regular, and $\forall \mathfrak{q} \in \operatorname{Spec} S$ minimal over $\mathfrak{m}$,

$$A_{\mathfrak{q}} \cong \operatorname{End}_{Z(A_{\mathfrak{q}})}(M),$$

where M is a reflexive $Z(A_{\mathfrak{q}})$-module.

Again consider $Y_1, \ldots, Y_n \subset \operatorname{Max} S$ non-intersecting algebraic sets. Set

$$I_i := I(Y_i), \quad R := \cap_{i=1}^{n} (k + I_i), \quad \mathfrak{m}_i := I_i \cap R,$$

and consider the 'noncommutative blowup' of A,

$$A := \operatorname{End}_R({}_R R \oplus \bigoplus_{i=1}^{n} \mathfrak{m}_i).$$

Theorem

- *A is a nonnoetherian NCR.*
- *If each I_i is a principal prime ideal of S, then A is nonnoetherian NCCR.*

Again consider

$$S = k[x, y] \quad \text{and} \quad R = k + xS.$$

Then

$$A = \mathrm{End}_R(R \oplus xS) \cong \begin{bmatrix} R & S \\ xS & S \end{bmatrix}$$

is a nonnoetherian NCCR of R.

Thank you!