An axiomatic divisibility theory for commutative rings

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Multiplication of integers led to the divisibility theory of integers and their prime factorization inspired probably Hensel to invent $p-$adic integers in 1897. p-adic numbers were generalized to the theory of real valuation by Kürschák in 1913 and further by Krull to one with values in ordered abelian groups in 1932. Values in Kürschák's and Krull's work are taken from the ordered field of reals and ordered abelian groups, respectively. These inventions open a question if there exists a valuation theory for certain rings possibly with zero-divisors and what would be their domain of values.

To solve this problem one has to learn divisibility theory in arbitrary rings carefully.

Divisibility

All groups, rings... are commutative with either 1 or $0 (= \infty)$.
Divisibility theory is the study of the relation

$$a \leq b \Longleftrightarrow a|b \Longleftrightarrow \exists c : b = ac \Rightarrow (a \leq b \leq a) \Longleftrightarrow aR = bR$$

The reverse inclusion is a *partial order* between principal ideals.

Definition

Divisibility theory of R is the multiplicative monoid S_R of principal ideals partially ordered by reverse inclusion. Sometimes divisibility theory can be understood as a multiplicative monoid of either all ideals or of finitely generated ideals partially ordered by reverse inclusion, too.

Object of study: narrowly Bezout and generally arithmetical rings.

Observations

$< x, y > \subsetneqq \mathbb{Z}[x, y]$ but $\mathrm{GCD}(x, y) = 1$. PIDs and UFDs have naturally partially ordered free groups as divisibility theory. Bezout domains have divisibility theory l.o. groups which are torsion-free. Divisibility theory of Boolean algebras is itself and one of (von Neumann) regular rings is a Boolean algebra of idempotents. This lead to an interesting question characterizing rings whose principal ideals have unique generators. All Boolean algebras and domains with the trivial unit group are such rings. All such rings are semisimple.

The answer: A theorem of Kearnes and Szendrei

Theorem 1

let $\mathfrak{Q}$ be the class rings whose principal ideals have unique generators and $\mathfrak{D} = \mathfrak{D}_1$ be the class of domains in $\mathfrak{Q}$.

1. *$\mathfrak{Q}$ is a quasivariety of rings axiomatized by the quasiidentity $(xyz = z) \to (yz = z)$. All such rings have trivial unit group and are $\mathbb{F}_2$-algebras.*
2. *$\mathfrak{D}$ consists of domains with trivial unit group.*
3. *$\mathfrak{Q} = SP(\mathfrak{D})$, i.e., the class of subrings of direct products of domains in $\mathfrak{D}$.*
4. *$\mathfrak{Q}$ is a relatively congruence distributive quasivariety.*
5. *The class of locally finite algebras in $\mathfrak{Q}$ is one of Boolean algebras which is the largest subvariety of $\mathfrak{Q}$.*

An example

There exists a ring with trivial unit group not contained in $\mathfrak{Q}$.

Example 1. Let R be an algebra over $\mathbb{F}_2$ generated by x, y, z subject to $xyz = z$, i.e., $R = A/I$ where $A = \mathbb{F}_2[X, Y, Z], I = A(XYZ - Z) = AZ \cap A(1 - XY) = I \cap I_2$. The isomorphisms $A/I_1 \cong F_2[X, Y], A/I_2 \cong F_2[X^{-1}, X][Z]$ shows that R has the trivial unit group but the ideal Rz has infinitely many generators $x^n z, y^n z, n \in \mathbb{N}$.

Quasiidentities and the unit group

For each $n \in \mathbb{N}$ let $\mathfrak{Q}_{|n}$ be the quasivariety of rings having the quasi-identity $xyz = z \Longrightarrow x^n z = y^n z = z$ and $\mathfrak{D}_{|n}$ the class of domains in $\mathfrak{Q}_{|n}$. The unit group of domains in $\mathfrak{D}_{|n}$ has an order a divisor of n. The unit group of rings in $\mathfrak{Q}_{|n}$ has an exponent a divisor of n.

Example 2. $R = \mathbb{Z}[x, y]/ < (x^2 + 1)y >; x^2 + 1, y$ irreducible in UFD $Z[x, y] \Rightarrow p \in \mathbb{Z}[x, y]$ generates a principal ideal of R with 4 generators if $y|p$ and $x^2 + 1 \nmid p$ otherwise this ideal has 2 generators if it is not trivial. An example of a ring in $\mathfrak{Q}_{|4}$ with two units $1, -1$.

Proposition 2

$R \in \mathfrak{Q}_{|n} \Longleftrightarrow$ *if the unit group of* $R/\operatorname{ann} a$ *has an exponent dividing* n *for all* $a \in R$. *In particular,* $R/\operatorname{ann} a \in \mathfrak{Q}_{|n}$.

Subdirect sum representation

Definition

$R \in \mathfrak{Q}_{|n}$ *(radically) subdirectly irreducible* $\Longleftrightarrow$ the intersection $\cap I \neq 0, I$ runs over $\{(\sqrt{I} =)I \lhd R \mid a \notin I \ \& \ R/I \in \mathfrak{Q}_{|n}\}$.

Proposition 3

A (semiprime) ring $R \in \mathfrak{Q}_{|n}$ *is a subdirect sum of (radically) subdirectly irreducible rings.*

Proof.

Using Zorn's Lemma and Proposition 2 to the set

$$\forall a \in R : \{a \notin (\sqrt{I} =)I \lhd R(\sqrt{0} = 0) \,\&\, R/I \in \mathfrak{Q}_{|n}\}.$$

$\square$

Radically subdirectly irreducible rings are domains

Proposition 4

Radically subdirectly irreducible rings D are domains.

Proof.

By assumption $\exists(0 \neq)a \in D$ such that $\sqrt{0} = 0$ is maximal in the set $\{(\sqrt{I} =)I \lhd R \mid a \notin I \ \& \ R/I \in \mathfrak{Q}_{|n}\}$. Furthermore, $\forall b \in D \ \sqrt{\operatorname{ann} b} = \operatorname{ann} b$ by $(br)^l = 0 \Rightarrow br = 0$. Proposition 3 implies $D/\operatorname{ann} b \in \mathfrak{Q}_{|n}$. Consequently, $\operatorname{ann} a = 0 \Rightarrow a \in \operatorname{ann} b \Rightarrow ab = 0 \Rightarrow D$ a domain with finite unit group of order an divisor of n. □

Subdirectly irreducible rings with zero-divisors 1

By Propositions 3, 4 s. i. rings with zero-divisors in $\mathfrak{Q}_{|n}, n > 1$ are more complicated, called shortly *subdirectly irreducible* (s.i.).

Proposition 5

$R \in \mathfrak{Q}_{|n}, n > 1$ *s. i.* $\Rightarrow \{\text{idempotents}\} = \{0, 1\}$, $\exists\ 0 = a^2 \neq a$ *s.t.* $P = \operatorname{ann} a = \{\text{all zero} - \text{divisors}\} \lhd \mathrm{R}$ *prime,* $D = R/P, \operatorname{char} D > 0, \dim_{R_P/P_P} aR_P = 1$. *If* $b^2 = 0 \Rightarrow nb = 0, (\operatorname{ann} P)^2 = 0, (\operatorname{char} D)a = 0.$

Subdirectly irreducible rings with zero-divisors 2

Proof.

$\exists (0 \neq) a \in R$ s.t. 0 maximal in $\{I \lhd R \mid a \notin I \ \&\ R/I \in \mathfrak{Q}_{|n}\}$.
$R/\operatorname{ann} a \in \mathfrak{Q}_{|n} \Rightarrow a \in \operatorname{ann} a$
$\Rightarrow a^2 = 0 \Rightarrow \exists (1+a)^{-1} \Rightarrow (1+a)^n = 1 + na = 1 \Rightarrow na = 0$.
$\forall b \in R \Rightarrow R/\operatorname{ann} b \in \mathfrak{Q}_{|n} \Rightarrow a \in \operatorname{ann} b \Longrightarrow ab = 0 \Rightarrow P =$ $\operatorname{ann} a \in \operatorname{Spec}(R)$ consists of all zero-divisors,
$D = R/\operatorname{ann} a \in \mathfrak{Q}_{|n} \Rightarrow D$ has $\leq n$ units. $b \notin P \Rightarrow nb \in P$ by $0 = b(na) = (nb)a \Rightarrow p = \operatorname{char} D | n$. $aP = 0 \Rightarrow aR_P$ a vector space over R_P/PR_P.
$\Rightarrow 0 = pa \in R_P \Rightarrow \exists u \notin P : upa = 0 \Rightarrow pa = 0$.
$D = R/\operatorname{ann} a \cong Ra \Rightarrow \dim aR_p = 1$.
$\operatorname{ann} P \subseteq \operatorname{ann} a = P \Rightarrow (\operatorname{ann} P)^2 = 0$.
$e^2 = e \in R \Rightarrow Re = \operatorname{ann}(1-e), R(1-e) = \operatorname{ann} e \Rightarrow a \in$ $Re \cap R(1-e) = 0 \Longrightarrow a = 0 \Rightarrow e \in \{1, 0\}$.
$b^2 = 0 \Rightarrow \exists (1+b)^{-1} \Rightarrow (1+b)^n = 1 + nb = 1 \Rightarrow nb = 0$. $\square$

Subdirectly irreducible rings with zero-divisors 3

Corollary 6

$\operatorname{ann} u = 0, u \in R \in \mathfrak{Q}_{|n}, \operatorname{char} R = 0$ *subdirectly irreducible,* $\Rightarrow u$ *is transcendental over* $\mathbb{Z}$ *or* $\exists\ \mathbb{Z}[lu],\ l \in \mathbb{N}$ *semiprime.* $\mathbb{Z}[lu]$ *is a finite direct sum of torsion-free domains which are imaginary quadratic extensions of* $\mathbb{Z}$ *with only 2, 4 or 6 units.*

Proof.

u algebraic $/\,\mathbb{Z} \Rightarrow \exists\, l \in \mathbb{N}$ such that a minimal polynomial q of lu is monic. The kernel of the canonical map $\mathbb{Z}[x] \to \mathbb{Z}[lu] : x \mapsto lu$ is $< q > \Rightarrow \mathbb{Z}[lu]$ torsion-free abelian group. By Proposition 5 $\mathbb{Z}[lu]$ is semiprime. q is a square-free product of irreducible polynomials. Consequently, $\mathbb{Z}[lu]$ is a finite subdirect sum of torsion-free domains. By Dirichlet units' theorem an algebraic extension of $\mathbb{Z}$ with finitely many units must be an imaginary quadratic extension of $\mathbb{Z}$, whence R has only 2, 4 or 6 units. □

Unit group and Mersenne primes

Example 3. $-m \in \mathbb{N} \Rightarrow T = \mathbb{Z}[\sqrt{m}]$ has 2 units if $m \notin \{-1, -3\}$, 4 units for $m = -1$ and $T = \mathbb{Z}[\theta], \theta = \frac{-1+\sqrt{-3}}{2}$ has 6 units.
$2 \in M \lhd T$ maximal (generated by $1 + i$ or $1 - \theta$ for $m = -1$ or $m = -3$, respectively,) then $R = T \rtimes T/M \in \mathfrak{Q}_{|n}, n \in \{2, 4, 6\}$.
Observation: $(-1)^2 = 1, R \in \mathfrak{Q}_{|(2n+1)} \Rightarrow \operatorname{char} R = 2$

Theorem 7

If the unit group is simple of order $p > 2$, then p is a Mersenne prime and R is a semiprime algebra over the field $\mathbb{F}_{p+1}$.

Proof.

$\operatorname{char} R = 2 \Rightarrow \sqrt{0} = 0$ by $a^2 = 0 \Rightarrow (1 + a)^2 = 1$ contradiction.
Maschke's theorem: $\mathbb{F}_2 G$ semisimple, a finite direct product of isomorphic fields having $p + 1$ elements and copies of $\mathbb{F}_2$.
$T = < G > \subseteq R$ as factor of $\mathbb{F}_2 G$ is a direct sum of the field $\mathbb{F}_{p+1}$ with copies of $\mathbb{F}_2$ whence p is a Mersenne prime. $\square$

Some further results

Corollary 8

$\mathfrak{Q}_{|p}$, p odd prime $\Rightarrow$ p Mersenne prime.

Proposition 9

Let $R \in \mathfrak{Q}_{|2}, U(R) = 1$, then $R \in \mathfrak{Q}$.

Proof.

R semiprime and $0 = 2 \in R$. $\forall a \in R : U(R/\operatorname{ann} a) = 1$ by $xy = 1(\operatorname{ann} a) \Rightarrow xya = a \Rightarrow x^2a = a \Rightarrow (ax + a)^2 = 0 \Rightarrow ax = x \Rightarrow x = 1(\operatorname{ann} a) \Rightarrow R$ a subdirect sum of $T \in \mathfrak{Q}_{|2}, U(T) = 1$. If T satisfies $\exists 0 \neq a \in T$ such that 0 maximal in $\{I \lhd T | T/I \in \mathfrak{Q}_{|2}, |U(T)| = 1\} \Rightarrow T$ a domain. □

Some open questions

$n \in \mathbb{N}$ a positive integer

1. Determine order n of the finite unit group
   $\Rightarrow$ all $\phi(m)$, $m \in \mathbb{N}$ can be an order of a unit group.
2. Determine rings with finite unit group.
3. Determine rings such that each principal ideal
   has at most n generators.
4. Determine all n appearing in Question 3.

Bezout monoids

Results of Clifford, Shores and Boschbach led to

Definition (Bosbach–Ánh–Márki–Vámos)

S *Bezout monoid*, shortly *B-monoid* $\iff a|b$ partial order such that

1. $\forall a, b \in S : \exists\, \mathrm{GCD}(a, b) = a \wedge b$,
2. $\forall\, a, b, c \in S : \; c(a \wedge b) = ca \wedge cb$,
3. S is *hypernormal*, i.e.,

$$\forall a, b : d = a \wedge b \,\&\, a = da_1 \Rightarrow \exists b_1 : b = db_1 \& a_1 \wedge b_1 = 1,$$

4. S has the greatest element 0.

Examples: Boolean algebras, positive cones of lattice-ordered groups endowed with the extra zero elements. B-monoids are distributive lattices.

Some remarks and properties

Idempotents form a Booolean algebra: $e^2 = e \Rightarrow d = e \wedge 0 = e$
$\Rightarrow \exists f \in S : 0 = df = ef, e \wedge f = 1 \Rightarrow f = f^2$.
B-monoids unify UFDs, Prüfer domains, semi-hereditary rings and arithmetical rings ($\Leftrightarrow$ ideal lattice is distributive).
In a series of papers Jensen showed

1. Finitely generated ideals form a B-monoid with reverse inclusion iff a ring is arithmetical.
2. Arithmetical ring semiprime iff
   $\mathrm{wdim} \leq 1 \Leftrightarrow \mathrm{Tor}_2(X, Y) = 0 \;\forall\; X,\; Y$
3. semihereditary rings are semiprime arithmetical rings but the converse is not true.

Semi-hereditary B-monoid $\Leftrightarrow \forall a \exists e = e^2 : a^\perp = \{x | ax = 0\}$.

Basic notions

Ideals is not appropriate to study B-monoids. The lack of addition forces to use further operation in the study of multiplication, the meet. Need combine both monoids and lattices: filters and m-prime filters. Addition is implicitly coded in the operation $\wedge$.

A subset F is a *filter* if it is closed under $\wedge$ and $b > a \in F \Rightarrow b \in F \Rightarrow$ filters are ideals and contain 0.

F is *m-prime*, simply *prime* if $ab \in F \Rightarrow a \in F$ or $b \in F$.

A subset C is an *m-cofilter* if $b, c \in C$, $a \leq b \Rightarrow\ a, bc \in C$.

A B-monoid is *semiprime* if 0 is the unique nilpotent element.

Congruences, homomorphisms of B-monoids 1

B-monoids is not closed under homomorphisms or subalgebras.

Proposition 10

To any filter F in a B-monoid S

$$x \cong y \Leftrightarrow \exists s \in S : x \wedge s = y \wedge s$$

defines a congruence whose factor S/F is a B-monoid.

Proposition 11

To a cofilter C in S $x \cong y \Leftrightarrow \exists s \in C : x \leq ys \,\&\, y \leq xs$ is a congruence whose factor S_C is the localization *of S at C, the congruence class of 0 is the filter $K = \{z | \exists s \in C : sz = 0\}$.*

Congruences, homomorphisms of B-monoids 2

Proposition 12

$\phi : S \to \phi(S)$ homomorphism between B-monoids$S, \phi(S)$. Then $C = \{s \,|\, \phi(s) = 1\}, F = \{s \,|\, \phi(s) = 0\} \Rightarrow C$ are m-cofilter, filter, respectivvely, $\phi(S)$ is isomorphic to the factor S_C/F_C where F_C is the image of F in S_C. All surjective homomorphisms between B-monoids can be obtained in this manner. Namely, if C is an m-cofilter and K is a filter then the relation

$$\Phi = \{(x, y) \in S \times S \mid s \in F, c \in C : x \wedge s \leq yc \;\&\; y \wedge s \leq xc\}$$

is a congruence on S whose 1-class is C and 0-class is generated by F in the obvious manner.

Structure of B-monoids

Basic properties of arithmetical rings by Jensen, Fuchs, Stephenson etc., can be carried over to B-monoids word by word. Any two prime filters are either coprime or comparable. Any prime filter has a unique minimal prime filter. $(\mathrm{m}-\mathrm{Spec}(S))\,\mathrm{Spec}(S)$ is the set of (minimal)prime filters with the Zariski topology given by sets $D_a, a \in S$ of prime filters not containing a as basis for open sets. A localization S_P of S by a prime filter P is a valuation B-monoid, i.e., any two elements are comparable, or equivalently, a local B-monoid. Both a Grothendieck and Pierce sheaf representation of rings can be adopted to B-monoids. In contrast to the local case, B-monoids with trivial idempotents are more complicated.

semiprime B-momoids

$\forall a \in S:\ S_a = \{b \,|\, a^\perp = b^\perp\}$

Proposition 13

In a semiprime B-monoid S, S_a is a cancellative subsemigroup closed under $\wedge$ and $\vee$, embedded in $(S/a^\perp)_1$. S_a is a positive cone of a l.o. group iff its partial order is natural. S is a disjoint union of cancellative lattice semigroups $S_a, a \in S$.

Semiprime B-monoids correspond to rings of weak dimension ≤ 1, i.e., to rings such that the second torsion products are 0. Although the class of semiprime B-monoids (semiprime arithmetical rings) is much bigger then one of semi-hereditary B-monoids (rings), it is harder to construct semiprime arithmetical rings which are not semihereditary.
The factor monoid of S by identifying $S_a, a \in S$ is unfortunately not a B-monoid but it is an interesting object of study.

Semihereditary B-monoids

Semihereditary B-monoids form a quite nice better understood class. $D_e = \{P \in \mathrm{m-Spec}(S) \mid e \notin P\}, e^2 = e \in S$ form an open basis of clopen sets for the Zariski topology. In semiprime B-monoids there exists a non-minimal prime filter of zero-divisors. In semi-hereditary B-monoids all prime filters of zero-divisors are minimal!

Proposition 14

A reduced B-monoid is semihereditary iff it satisfies one of the following equivalent properties.

1. S *is semihereditary.*
2. $\forall a \ \exists b \ : a^{\perp} = bS$.
3. $\forall a \, \exists b \ : a \wedge b \in S_1$.
4. *The minimal spectrum is compact.*

B-monoids with one minimal prime filter 1

Proposition 15

A B-monoid with finitely many minimal prime filters is a finite direct sum (i.e., finite Cartesian product) of B-monoid with one minimal prime filter.

Proposition 16

S a B-monoid; M a smallest minimal m-prime filter, $T = S \setminus M$, $Z = \{x \in S \,|\, \exists s \notin M : sx = 0\} \subseteq M; N = M \setminus Z \Rightarrow ZM = 0$; $t < n < z \;\forall\; t \in T,\, n \in N,\, z \in Z$; and T non-negative cone of l.o. group G.

Classical localization $T^{-1}S$ inverting T is not B-monoid but its *divisibility*, the monoid of principal filters order-isomorphic to S_M sending $T = S \setminus M \mapsto 1$
Crucial examples: factors of $\mathbb{Z} + x\mathbb{Q}[x]$ by $x^n\mathbb{Q}[x]$ or by $x^n\mathbb{Q}[x] + x^{n-1}\mathbb{Z}[x]$, $n > 1$ and their divisibility theory.

B-monoids with one minimal prime filter 2

Notation: $X^\bullet = X \dot\cup 0; \Sigma = S_M = \{\alpha = a^\sigma = T^{-1}Sa \mid a \in S\}$

$$S_a = S_\alpha = \{b \in S \mid b^\sigma = \alpha = a^\sigma\} \Rightarrow S_1 = T,\, S_0 = Z \Rightarrow$$

$s \in N \Rightarrow S_s \cong G \Rightarrow G$ acts on N. $x^\sigma < y^\sigma \Rightarrow x < y$.

Proposition 17

$xy = y \notin Z \Rightarrow x = 1$, $Y = S \setminus Z \to T^{-1}S$ injective. The filter of $T^{-1}S$ generated by N is exactly $N^\bullet$. $G =< T, T^{-1} >$ acts on N, $a \in N \Rightarrow Ga = S_a$. Divisibility monoid of $T^{-1}S$ is Σ, $S : x^\sigma < y^\sigma \Longrightarrow x < y$. $T^{-1}S$ is $X^\bullet; X = G \dot\cup N$, and $Z \neq M \Rightarrow Z$ a factor of G by an appropriate filter.

Structure of B-monoids with one minimal prime filter 3

Theorem 18

As above $M \neq Z \Rightarrow \ \exists\, A$ l.o. group; filters $B_\infty \subseteq C_\infty \subseteq P = \{g \in A \mid g \geq 1\}$:

1. $P =< P \setminus C_\infty >$
2. *Rees factors $P/C_\infty \cong S/Z \Rightarrow S \cong P/C_\infty$ if $Z = 0$;*
3. $Z \neq 0 :\ S \cong P/B_\infty$ *by* $a \sim b \Longleftrightarrow \exists\, c \in B_\infty : a \wedge c = b \wedge c.$

(1) and (2) determine P, A uniquely up to isomorphism fixing $S \setminus Z$ elementwise by identification of $S \setminus Z$ with $P \setminus C_\infty$.

Clifford's result: local B-monoids are Rees factors of positive cones of ordered abelian groups.

Theorem 19

S B-monoid; M unique minimal m-prime filter, $T = S \setminus M$, $Z = \{s \,|\, \exists\, t \notin M : ts = 0\}$ factor of the quotient group of $T \Rightarrow S$ factor of nonnegative cone of a l.o. group.

Corollary 20

As above, $|\,\Sigma\,| > 2 \Rightarrow S$ *factor of nonnegative cone of a l.o. group.*

Theorem 21

S *B-monoid;* M *unique minimal m-prime filter s.t.* $M = Z \neq 0$*. If the filter generated by all* $a^{\perp}(0 \neq a \in M)$ *proper, then* S *a factor of non-negative cone of a l.o. group. More generally, if* $I \lhd S$ *m-prime filter in a B-monoid* S*,*
$K = \{s \in S \,|\, \exists t \notin I : ts = 0\} \Rightarrow S/K$ *factor of nonnegative cone of a l.o. group.*

Basic problem

It is a basic problem to represent B-monoids as divisibility theory of appropriate rings. This lead to several new interesting classes of rings. Firstly, Krull constructed to any ordered abelian group a domain (field) whose divisibility theory is isomorphic to this group. Secondly, Ohm, Jaffard constructed independently to any l.o. abelian group a domain (field) whose divisibility theory is this group. The case of local B-monoids is settled by an observation of Shores, although it can be constructed as a localization of the 0-constructed monoid algebra at the set of primitive elements. The main aim is the construction a ring with an arbitrary predescibed B-monoid as its divisibility theory.

Representation of semi-hereditary B-monoids

Dedekind and Prüfer domains lead semi-hereditary rings which are semiprime.

Theorem 22 (Ánh–Siddoway)

To any semi-hereditary Bezout monoid S there is a semi-hereditary Bezout ring R whose divisibility theory is order-isomorphic to S.

S_e, $e^2 = e \in S$ positive cones in l.o. groups, $S_1 = S_e \wedge (e^{\perp})_1$; D_e clopen sets of $X = \mathrm{m}-\mathrm{Spec}(S)$. χ_e the characteristic function on D_e. There are two constructions of semi-hereditary Bezout algebras with divisibility theory isomorphic to S. Let L arbitrary field and

$$A = (LS_1)_P,\ P = \{\sum_{i=1}^{n} k_i s_i \mid 0 \neq k_i \in L,\, s_i \in S_1,\ \bigwedge_{i=1}^{n} s_i = 1\}$$

K the field of fractions of A with the discrete topology.

The construction 1

S_1 is the divisibility theory of A. To $e^2 = e \in S$ put $P_e = \{e \wedge t \mid t \in (e^{\perp})_1\} \subseteq S_1 \subseteq A$ and $A_e = A_{P_e}$. One has $A_0 = A, A_1 = LS_1, A_e \subseteq A_g$ if $eg = g$. $e_1^{\perp}$ is the divisibility theory of A_e. $C_K(X) = \{\chi : X \to K \mid \chi \text{ continuous}\} \Rightarrow \chi$-s are linear combinations $\sum_{i=0}^{n-1} a_i \chi_{e_i}, a_i \in K$ where e_i pairwise orthogonal and $e_i = 1$, with $a_0 = 0$. Note that D_{e_0} can be empty. The subring R of $C_K(X)$ of all linear combinations $\sum_{i=0}^{n-1} a_i \chi_{e_i}, a_i \in A_{e_i}$ is a semihereditary algebra over K with divisibility theory isomorphic to S.

The construction 2

A factor of the 0-contracted monoid algebra LS subject to $e + e' = 1, e^2 = e \in S \Rightarrow$

$$A = \{p = \sum_i p_i \mid p_i \in LS_{e_i}, e_i \text{pairwise orthogonal}\}$$

$p \in A$ is *primitive* if $\wedge_i e_i = 1$ and all p_i are primitive in LS_{e_i}. The set of primitive elements of A is multiplicative closed and is a subset of non-zero-divisors of A whence R is just the localization of A at the set of primitive elements.

Observations

In contrast to lattice-ordered groups, this construction of R is not a free construction unless $S = S_1$, i.e., except the case of domains. Since Booolean algebras B has the trivial unit group, it is reasonable at least in our theory to consider the trivial group torsion-free.

The case of a unique minimal prime filter

Bezout ring with one minimal prime ideals including Bezout domains: simplest examples for not necessarily semiprime arithmetical rings having compact minimal spectrum.

Theorem 23 (Ánh–Siddoway)

S B-monoid with one minimal m-prime filter $\Rightarrow$ $\exists$ a Bezout ring having S as its divisibility theory.

A construction is based on the description of S in Theorem 18. In the case that S is a factor of a positive cone in a l.o. group, then a ring is a factor of the Bezout domain associated to this group. In the remainder case, a ring is a trivial extension a Bezout domain associated to the group determined by T and its Bezout module constructed similarly one of non-standard uniserial modules using direct limits.

Factors of Bezout rings

Theorem 24

If R is a Bezout ring with one minimal prime ideal I such that the localization R_I is not a field, then the divisibility theory S_R of R is a lattice factor of a positive cone of a lattice-ordered group.

This result suggests several interesting open problems. One of them is find a counter-example of a Bezout monoid with one minimal m-primer filter which is not a factor of a positive cone of a group. This corresponds to Kaplansky's problem on factors of valuation domains. Even more important is the description of factors of Bezout domains. Note that factors of polynomial rings which are UFDs, yield all commutative algebras!

Final remarks, open questions 1

1. Homological theory for semiprime B-monoids.
2. Structure of semiprime B-monoids.

Semi-hereditary rings are homologically well-understood, but what are they? Semi-hereditary Bezout monoids could provide new examples and one can construct semi-hereditary Bezout monoids. No satisfactory structure theory for semiprime Bezout monoids which correspond to rings of weak dimension at most 1. Although there must be much such rings but it seems that there are very few examples for them! They seem to be exceptional although there are no reasons for that.

3. L.o. groups provide common frame for UFDs and Bezout domains. A vague question: to which (not necessarily arithmetical) rings can be naturally associated B-monoids ?

Final remarks, open questions 2

Representation of B-monoids is still open. The main difficulty in a construction is the fact that in contrast to classical cases of domains, addition is not free although it is encoded partly possibly in minimal spectra of Bezout monoids! Our representation theorems are constructed always by ad-hoc construction. Sheaves, mainly Peirce sheaves could come in action. It would be good assistance to get closed description of Peirce sheaf representation of semiprime B-monoids as well as description of indecompasable B-monoids. It is an open question whether there is an indecomposable B-monoid with infinitely many minimal prime filters.

Thank You for Your Attention