PERINORMAL RINGS WITH ZERO DIVISORS

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Regular ideal

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Definition

An **overring** of a ring A is a ring between A and its total quotient ring.

Regular and large quotient ring

Definition

Let A be a ring with total quotient ring K and P be a prime ideal of A. Then
(1) $A_{[P]}=\{x \in K | xy \in A \text{ for some } y \in A - P\}$ is called the **large quotient ring**.
(2) $A_{(P)}=A_S$ where $S = (A - P) \cap Reg(A)$ is called the **regular quotient ring**.

We have $A \subseteq A_{(P)} \subseteq A_{[P]} \subseteq K$.

Going down

Definition

A ring extension $A \subseteq B$ satisfies **going down (GD)** if whenever $P \subseteq Q$ are prime ideals of A and $Q' \in \operatorname{Spec} B$ with $Q' \cap A = Q$, there exists some $P' \in \operatorname{Spec} B$ with $P' \subseteq Q'$ and $P' \cap A = P$.

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We say that an overring B of A is a **GD-overring** if $A \subseteq B$ satisfies going down.

Flat overrings

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Let A be a ring and $B \subseteq C$ overrings of A. (a) The extension $A \subseteq B$ is flat if and only if $A_{[M \cap A]} = B_{[M]}$ for all maximal ideals M of B.
(b) If $A \subseteq B$ and $B \subseteq C$ are flat, then $A \subseteq B$ is flat.
(c) If $A \subseteq C$ is flat, then $B \subseteq C$ is flat.
(d) If $A \subseteq B$ is flat and integral, then $A = B$.

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Valuation ring

Definition

A **(Manis) valuation** on a ring A is a surjective map $v : A \to G \cup \{\infty\}$ where $(G;+)$ is a totally ordered abelian group, such that;

1. $v(xy) = v(x) + v(y)$, for all x and y in A.
2. $v(x+y) \geq min\{v(x), v(y)\}$, for all x and y in A.
3. $v(1) = 0$ and $v(0) = \infty$.

Valuation ring

Theorem (Manis 1969)

Let A be a ring with total quotient ring K and let M be a prime ideal of A. Then the following conditions are equivalent.
(a) If B is an overring of A having a prime ideal N such that $N \cap A = M$, then $A = B$.
(b) If $x \in K - A$, there exists $y \in M$ such that $xy \in A - M$.
(c) There exists a valuation v on K such that $A = \{x \in K \mid v(x) \geq (0)\}$ and $M = \{x \in K \mid v(x) > 0\}$.

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A pair (A, M) satisfying the equivalent conditions of above theorem is called a **valuation pair** and A is called a **valuation ring** (on K). If G is the group of integers, then A is called a discrete valuation ring (DVR).

Prüfer rings

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Theorem (Griffin 1970)

Let A be a ring with total quotient ring K. Then the following assertions are equivalent:

1. *A is a Prüfer ring;*
2. *For each $M \in Max(A)$, $(A_{[M]}, [M]A_{[M]})$ is a valuation pair;*
3. *Each overring of A is A-flat.*

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Remark

There exists valuation rings that are not Prüfer.

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- *Each overring of a Marot ring is Marot ring.*
- *If a ring A is Marot, then A satisfies the following condition: (*) $[P]A_{[P]}$ contains all regular nonunits of $A_{[P]}$ for every regular prime ideal P.*
  Condition $(*)$ was considered by Boison.

(M. Boisen, *The containment property for large quotient rings*, 1973.).

Krull rings

Definition

A Marot ring A is called a **Krull ring** *if each regular ideal is t-invertible.*

Theorem

If S a multiplicatively closed subset of a Krull ring A, then $A_{(S)}$ is a Krull ring.

Perinormal rings

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Definition

A ring A is called **perinormal** if whenever B is an overring of A such that $A \subseteq B$ satisfies going down, it follows that B is A-flat.

Fact: Every Prüfer ring is perinormal. (A. Rani and T. Dumitrescu, *Perinormal rings with zero divisors* (to appear in J. Commut. Algebra).)

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Let A be a perinormal ring and B a flat overring of A. Then B is also perinormal.

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Perinormality is not a local property in the classical sense, that is, if A_M is perinormal for each maximal ideal M of a ring A, then A is perinormal. The converse is not true.

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For example:

Example

Let (B, M) be a local one-dimenional domain which is not integrally closed, $C = B[X, Y]/X(M, X, Y)$ with X, Y are indeterminates and $A = C_N$ where $N = (M, X, Y)$ is local with the maximal ideal NA_N. Since every non unit is a zero divisor $(M.X = 0)$, $Tot(A) = A$. Hence A is perinormal.
On the other hand, if $P = (M, X)A$, then $A_P \simeq B(Y)$ is a one-dimenional domain which is not integrally closed, hence A_P is not perinormal.

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Definition

A ring A is called a P-**ring** if $(A_{[Q]}, [Q]A_{[Q]})$ is a Manis valuation pair for every $Q \in Ass_A(K/A)$.

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Theorem

If A is a P-ring satisfying condition $(*)$, then A is perinormal.

(A. Rani and T. Dumitrescu, *Perinormal rings with zero divisors* (to appear in J. Commut. Algebra).)

Perinormal rings

Corollary

Let A be a Marot ring. In the following list, every assertion implies the next one.

(a) A is Noetherian and integrally closed.
(b) A is a Krull ring.
(c) A is a PvMR.
(d) A is a P-ring.
(e) A is perinormal.

Noetherian perinormal rings

Definition

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The following theorem extends a result due to McAdam for rings with zero divisors. Our proof is just an adaptation of the original proof.

Theorem (McAdam 1972)

Let A be a Noetherian ring. If B is an integral GD-overring of A, then the non-minimal regular prime ideals of A are unibranched in B.

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Theorem (Epstein and Shapiro 2016)

Let D be a Noetherian domain with integral closure D'. Assume that $D_{P \cap A}$ is a DVR for every height one prime ideal P of D'. The following are equivalent.
(a) D is perinormal.
(b) For each $P \in Spec(D)$, D_P is the only ring C between D_P and its integral closure such that $D_P \subseteq C$ is unibranched.

(N. Epstein and J. Shapiro, *Perinormality-a generalization of Krull domains, J. Algebra*, 2016.).

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Theorem

Let A be a Noetherian ring with integral closure A' and total quotient ring K. Assume that $(A_{(P\cap A)}, PA_{(P\cap A)})$ is a Manis valuation pair (in K) for every minimal regular prime P of A'. The following assertions are equivalent.
(a) A is perinormal.
(b) Whenever P is a prime ideal of A and B is an integral unibranched overring of $A_{(P)}$, we have $A_{(P)} = B$.

(A. Rani and T. Dumitrescu, *Perinormal rings with zero divisors* (to appear in J. Commut. Algebra).)

My Questions

1. A ring A is perinormal if and only if $A_{[M]}$ is perinormal for all M in $Max(A)$?
2. Is a non Prüfer valuation ring is perinormal?

Thank you!