ALMOST PERFECT COMMUTATIVE RINGS

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Abstract. We present the development of the theory of almost perfet commutative rings, from their birth in solving a module theoretical problem, passing to the first basic results for almost perfect domains, showing the blossom of the theory using the tool of cotorsion pairs, and finally arriving to our days with the appearance on the stage of zero-divisors. In the last part of the talk we present results recently obtained in a joint paper with Laszlo Fuchs, focusing on the structural properties of the rings.

SUMMARY

1. The birth of almost perfect domains

2. Ring theoretical results

3. Examples of almost perfect domains

4. Module theoretical results and the tool of cotorsion pairs

5. The appearance of zero-divisors

6. Examples of almost perfect rings with zero-divisors

1. The birth of almost perfect domains

Almost perfect domains came into the world in 2002 when, in a paper with Silvana Bazzoni, we answered the following question posed by Jan Trlifaj at the conference that I organized in Cortona in 2000:

when all modules over an integral domain R have a strongly flat cover?

A module S is <u>strongly flat</u> if $\mathrm{Ext}_R^1(S,C) = 0$ for all Matlis-cotorsion modules C.
The module C is <u>Matlis-cotorsion</u> if $\mathrm{Ext}_R^1(Q,C) = 0$ (Q is the field of fractions of R).
So S is strongly flat if every short exact sequence of R-modules

$$0 \to C \to X \to S \to 0$$

with C Matlis-cotorsion splits.

Strongly flat modules S are direct summands of modules which are extension of a free module by a divisible torsionfree module, clearly, such a module is flat:

$$0 \to \bigoplus R \to S \oplus X \to \bigoplus Q \to 0$$

A strongly flat module S is a <u>strongly flat cover</u> for a module M if

* there exists a map $f : S \to M$ such that for every map $f' : S' \to M$, with S' strongly flat, there exists a map $g : S' \to S$ satisfying: $f' = f \circ g$, and

** if an endomorphism $g : S \to S$ satisfies $f = f \circ g$, then g is an automorphism.

Trlifaj's question is inspired by the famous 1960 result by H. Bass:

every module over a ring R has a projective cover iff R is perfect.

Our answer with Silvana was that the following conditions are equivalent:

- *every module over a domain R has a strongly flat cover*
- *every flat module is strongly flat*
- *R is an almost perfect domain, i.e., every proper quotient of R is perfect.*

Recall the Bass' characterizations of perfect commutative rings.

THEOREM. (Bass, 1960) *For a commutative ring R TFAE:*

1) R is perfect, i.e., flat modules are projective;

2) The class of projective modules is closed under taking direct limits;

3) All R-modules have a projective cover;

4) R satisfies the descending chain condition on principal ideals;

5) R is a finite direct product of local rings with T-nilpotent maximal ideals;

6) R is semi-local and semi-artinian.

REMARK 1. The characterizations in 1), 2) and 3) are module theoretical; they are enlightened by the result, proved only in 2001 by Bican-El Bashir-Enochs, that all modules over any ring admit a flat cover.

REMARK 2. The characterizations in 3), 4) and 5) are ring theoretical.
It appears the crucial notion of T-nilpotency of an ideal I, which means that:

given a sequence of elements $\{a_n\}_{n>0}$ in I, there exists an index k such that $a_1 a_2 \dots a_k = 0$.

Recall also that R semi-local means with finite maximal spectrum, and semi-artinian that R/I has non-zero socle for every ideal I.

REMARK 3. In 1969, nine years after Bass, Smith introduced local TTN-domains i.e., domains with topologically T-nilpotent maximal ideal P, that is:

given a sequence of elements $\{a_n\}_{n>0}$ in P and $0 \neq b \in R$, there exists an index k such that $a_1 a_2 \ldots a_k \in bR$.

These domains are the local case of APD's *(ante-litteram, 33 years in advance)*.

2. Ring theoretical results

The rings we consider are always commutative. The first important fact on almost perfect rings is that we can disregard rings with zero-divisors.

PROPOSITION. *If an almost perfect ring R is not a domain, then R is perfect.*

Sketch of the proof. R is 0-dimensional since, if $0 \neq L$ is a prime ideal, then R/L is a perfect domain, hence a field. It is possible to reduce to the local case. For R a 0-dimensional local ring with maximal ideal P, one can show that, if R/aR is perfect for some $a \in P$, then R/a^2R is also perfect. Since P is the nilradical of R, some power of a is zero, so R is perfect. ///

In view of the preceding proposition, generalizations to rings with zero-divisors seem out of question. However, asking that the quotients of R are perfect not modulo all non-zero ideals, but only modulo regular ideals, we will obtain a significant generalization, the main topic of this talk discussed in Section 5.

The crucial properties of APD's are given in the following

THEOREM. *A domain is an APD if and only if it is h-local and each localization at a maximal ideal is a TTN-domain. Such a ring is a 1-dimensional Matlis domain.*

REMARKS. 1) h-locality intervenes passing from local to global, as in case of almost maximal domains.

2) For 1-dimensional domains h-locality is equivalent to the finite character property, i.e., every non-zero ideal is contained in finitely many maximal ideals.

3) A Matlis domain is defined by the property that p.d.Q = 1; 1-dimensional h-local domains are necessarily Matlis.

Looking at <u>local</u> almost perfet domains R, a first relevant distinction concerns the behaviour of their maximal ideal P.

A stronger property for P with respect to the topological T-nilpotency is that of being <u>almost nilpotent</u>, i.e., for every ideal $I \neq 0$ there exists a positive integer n depending on I such that $P^n \leq I$. So: P almost nilpotent ==> R APD.

PROPOSITION. *For a local APD R with maximal ideal P, TFAE:*

1) P is almost nilpotent

2) The Loewy length of Q/R is ω.

So an interesting question is:

which (limit) ordinals are admissible as Loewy length of Q/R for an APD R whose maximal ideal P is not almost nilpotent?

At the moment a complete answer is not available. However, by means of a very elaborate construction, it is possible to provide concrete examples of APD's R such that Q/R has Loewy length $\omega \cdot n$ for any positive integer n [S-Zanardo, 2004]. I do not known whether we can pass from $\omega \cdot n$ to $\omega \cdot \omega$.

Here are some results connecting APD's with other classes of domains:

- 1-dimensional Noetherian domains are APD's
- a coherent APD is Noetherian
- a Pruefer domain is an APD if and only if it is Dedekind
- if R is a local APD with maximal ideal P, then it is a DVR iff P is principal, it is Noetherian iff P/P^2 is finitely generated.

At this stage it is worth providing some concrete examples of APD's.

3. Examples of almost perfect domains

EXAMPLE 1. A family of examples of APD's R is obtained from two fields $F < K$:

$$R = F + XK[[X]]$$

i.e., R consists of the power series over K with constatnt term in F.

R is local with almost nilpotent maximal ideal $P = XK[[X]]$, hence R is an APD.

- R Noetherian <==> $[K : F] < \infty$
- R integrally closed <==> F algebraically closed in K

Particular examples:

- $R_1 = \mathbf{R} + X\mathbf{C}[[X]]$ is Noetherian not integrally closed
- $R_2 = \mathbf{A} + X\mathbf{C}[[X]]$ is integrally closed not Noetherian (**A**=algebraic numbers)
- $R_3 = \mathbf{Q} + X\mathbf{C}[[X]]$ is neither Noetherian nor integrally closed

EXAMPLE 2. (Smith 1969)

Start with R local APD with maximal ideal P and field of quotients Q and let F be a field containing Q. Let $\alpha \in F$ be integral over R, root of a monic polynomial

$$f(X) = X^{n+1} + r_n X^n + \dots + r_1 X + r_0 \in R[X]$$

of minimal degree > 1. Then:

$$R[\alpha] \text{ is an APD} \Longleftrightarrow r_i \in P \text{ for all } i.$$

In this case, $R[\alpha]$ is local with maximal ideal $M = P + \alpha R + \alpha^2 R + \dots + \alpha^n R$.

- $R[\alpha]$ Noetherian <==> R Noetherian
- M almost nilpotent <==> P almost nilpotent

Particular example:

- $R = \mathbf{Z}_p[p\sqrt{p}]$ the minimal polynomial of $\alpha = p\sqrt{p}$ is $f(X) = X^2 - p^3$.

EXAMPLE 3. A non-Noetherian APD which fails to be local is

$$R = \mathbf{Q} + X\mathbf{C}[X]$$

i.e., the complex polynomials with rational constant term.

- Max(R) = { $P = X\mathbf{C}[X]$; $P_a = (1-aX)R : 0 \neq a \in \mathbf{C}$ } has cardinality the continuum
- R_{P_a} is a DVR for all $0 \neq a$
- R_P is a non-Noetherian local APD.

4. Module theoretical results and the tool of cotorsion pairs

The property of a domain of being almost perfect has a relevant impact on the structure of <u>torsion modules</u>, <u>flat modules</u> and <u>divisible modules</u>.

THEOREM 1. R is an almost perfect domain if and only if it is h-local and one of the following equivalent conditions hold:

(i) every torsion cyclic module R/I contains a simple module

(ii) Q/R is semi-artinian

(iii) every torsion module T is semi-artinian

If R is almost perfect, then $Q/R \cong \bigoplus_{P \in \mathrm{Max}R}(Q/R_P)$ and Q/R_P is semi-artinian for all P

If T is a torsion module, then $T \cong \bigoplus_{P \in \mathrm{Max}R} T_P$ and T_P is semi-artinian for all P

THEOREM 2. For an integral domain R, TFAE:

(i) R is almost perfect

(ii) every flat module is strongly flat

(iii) the class of strongly flat modules is closed under direct limits.

If these conditions hold, then for an R-module M the TFAE:

(a) M is flat

(b) the completion of M in the R-topology is a summand of the completion o a free module

(c) $M \otimes_R (Q/R)$ is isomorphic to a summand of a direct sum of copies of Q/R.

If R is local, then “a summand of” can be cancelled in (b) and (c).

If R is almost perfect, then a module D is divisible if and only if PD = D for every maximal ideal P. Furthermore, a homological characterization of D id available.

Recall that the weak (or flat) dimension w.d.M of a module M is similar to the projective dimension p.d.M, with flat modules replacing projective module.

A module W is <u>weak injective</u> if $\mathrm{Ext}_R^1(F,W) = 0$ for every module F with w.d.F ≤ 1. Such a module W is divisible, since divisible modules D are characterized by the property that $\mathrm{Ext}_R^1(R/I,D) = 0$ for all invertible ideals I (i.e., p.d.R/I = 1).

THEOREM 3. An integral domain R is is almost perfect if and only if every divisible module is weak injective.

The previous results can be summarized in a concise way using cotorsion pairs, a tool that I introduced in 1979 under the name of cotorsion theories.

My goal in the late '70's was to generalize cotorsion Abelian groups C, defined by the property that $\mathrm{Ext}_Z^1(\mathbf{Q},C) = 0$. In the survey paper: "*Abelian Group Theory in Italy*" I called the cotorsion theories the "sleeping beauties", because they slept until 2000, when Goebel and Shelah solved in the paper "*Cotorsion theories and splitters*" a problem that I posed in 1979.

If $\mathcal{A}$, $\mathcal{B}$ are two classes of R-modules, we say that $(\mathcal{A},\mathcal{B})$ is a <u>cotorsion pair</u> if

$\mathrm{Ext}_R^1(A,B) = 0$ for all $A \in \mathcal{A}$, $B \in \mathcal{B}$, and $\mathcal{A}$, $\mathcal{B}$ are the largest classes with this

property; $\mathcal{B}$ is the right Ext-orthogonal of $\mathcal{A}$ and $\mathcal{A}$ is the left Ext-orthogonal of $\mathcal{B}$.

Examples of cotorsion pairs over any ring R are:

$(\mathcal{P}, \mathrm{Mod}(R))$	$\mathcal{P}$ = projective modules		
$(\mathrm{Mod}(R), \mathcal{I})$	$\mathcal{I}$ = injective modules		
$(\mathcal{SF}, \mathcal{M})$	$\mathcal{SF}$ = strongly flat modules	,	$\mathcal{M}$ = Matlis-cotorsion modules
$(\mathcal{F}, \mathcal{C})$	$\mathcal{F}$ = flat modules	,	$\mathcal{C}$ = Enochs-cotorsion modules
$(\mathcal{P}_1, \mathcal{D})$	$\mathcal{P}_1$ = modules of p.d. ≤ 1	,	$\mathcal{D}$ = divisible modules
$(\mathcal{F}_1, \mathcal{WI})$	$\mathcal{F}_1$ = modules of w.d. ≤ 1	,	$\mathcal{WI}$ = weak injective modules

The preceding characterizations of almost perfect domains via different classes of modules can be expressed using cotorsion pairs.

MAIN HOMOLOGICAL THEOREM. For a commutative integral domain R TFAE:

1) R is almost perfect;

2) $(\mathcal{SF}, \mathcal{M}) = (\mathcal{F}, \mathcal{C})$, equivalently, flat modules are strongly flat, or Matlis-cotorsion modules are Enochs-cotorsion;

3) $(\mathcal{P}_1, \mathcal{D}) = (\mathcal{F}_1, \mathcal{WI})$, equivalently, w.d.M ≤ 1 implies p.d. M ≤ 1, or divisible modules are weakly injective.

Over arbitrary commutative rings, $(\mathcal{P}_1, \mathcal{D})$ is not a cotorsion pair; $\mathcal{P}_1$ must be replaced by $\mathcal{CS}$, the class consiting of summands of modules with a filtration of cyclically presented modules.

The two classes $\mathcal{P}_1$ and $\mathcal{CS}$ coincide if and only if f.dim Q = 0 (f.dim is the little finitistic dimension) where Q is the classical ring of quotients of R, in particular, if R is a domain or a perfect ring (see Bazzoni-Herbera 2009).

The results exposed up to now have been obtained between 2002 and 2010. Researches on almost perfect rings stopped until 2016.

We omitted here, because of limited space and time, to discuss the connection of almost perfect rings with the existence of covers and envelopes, which produces other interesting characterizations of almost perfect domains.

5. The appearance of zero-divisors

In 2016 Laszlo Fuchs proposed to call almost perfect a commutative ring R such that R/rR is perfect for every <u>regular</u> element $r \in R$, i.e, for every $r \in R^x$.

Obviously, an APD is almost perfect in this sense, since every non-zero element of a domain is regular.

Note that, given a module M over a ring R with zero-divisors

- an element $x \in M$ is <u>torsion</u> if $rx = 0$ for some $r \in R^x$
- M is <u>divisible</u> if $M = rM$ for all $r \in R^x$.

So the above new definition is quite natural in the presence of zero-divisors.

However, if one looks for an extension to this setting of the equivalence:

$$\text{R almost prfect} \Longleftrightarrow (\mathcal{P}_1, \mathcal{D}) = (\mathcal{F}_1, \mathcal{WI})$$

(thus requiring $(\mathcal{P}_1, \mathcal{D})$ a cotorsion pair, so f.dim Q = 0) a new crucial condition comes up.

PROPOSITION. If for a commutative ring R the equality $(\mathcal{P}_1, \mathcal{D}) = (\mathcal{F}_1, \mathcal{WI})$ holds, then the classical ring of quotients Q of R is a perfect ring.

Proof. We show hat a flat Q-module F is projective. F is a flat R-module too, since the injection $R \to Q$ is a ring epimorphism, hence $F \in \mathcal{F} \subseteq \mathcal{F}_1 = \mathcal{P}_1$. Take any Q-module D; then $D \in \mathcal{D}$, hence $\mathrm{Ext}_R^1(F,D) = 0$, since $(\mathcal{P}_1, \mathcal{D})$ is a cotorsion pair. The homological formula: $\mathrm{Ext}_Q^1(F\otimes_R Q, D) \cong \mathrm{Ext}_R^1(F, \mathrm{Hom}_Q(Q,D)) = \mathrm{Ext}_R^1(F, D) = 0$, due to Fuchs-Lee (Comm. Alg. 37 (2009), 923-932) shows that $F \cong F\otimes_R Q$ is a projective Q-module.

The rings R such that Q is perfect have been characterized by Gupta in 1970.

LEMMA (Gupta). The quotient ring Q of the commutative ring R is perfect if and only if: (i) the nilradical N of R is T-nilpotent

(ii) a regular element of R/N has a regular representative in R

(iii) R/N is a Goldie ring.

Recall that a Goldie ring has finite Goldie dimension and the ACC on annihilators of subsets. So we are led to the following

DEFINITION. A commutative ring R is <u>almost perfect</u> if its classical ring of quotients Q is perfect and R/rR is a perfect ring for every $r \in R^x$.

Thus, if R is an almost perfect ring, then its nilradical N is T-nilpotent and R/N is a Goldie ring.

With this definition at hand one can extend *verbatim* (but with some effort) from APD's to almost perfect rings the main homological theorem.

THEOREM. For a commutative ring R with perfect ring of quotients TFAE:

(1) R is almost perfect

(2) flat modules are strongly flat

(3) Matlis-cotorsion modules are Enochs-cotorsion

(4) module of weak-dimension ≤ 1 have projective dimansion ≤ 1

(5) the cotorsion pairs $(\mathcal{P}_1, \mathcal{D})$ and $(\mathcal{F}_1, \mathcal{WI})$ coincide

(6) divisible modules are weak injective

(7) h-divisible modules are weak injective

(8) epimorphis images of weak-injective modules are weak-injective.

From now on we will focus on the structure of APR's.

Recall that a <u>subdirect product</u> of a family of rings $\{R_i\}_{i\in I}$ is a ring R embedded into the direct product:

$$\varepsilon \ : \ R \rightarrowtail \prod_{i\in I} R_i$$

in such a way that $\pi_j \bullet \varepsilon$ is surjective for every canonical surjection $\pi_j : \prod_{i\in I} R_i \rightarrow R_j$

We will consider only <u>finite subdirect products</u>, i.e., the index set I is finite.

To prove the structural theorems, we need the following result.

LEMMA. A ring R which is a finite subdirect product of perfect rings R_i $(1 \leq i < n)$ is perfect. If each ring R_i is local with maximal ideal M_i, then R is isomorphic to $\prod_i R_i$ if and only if $\mathrm{Ker}(\pi_j \bullet \varepsilon)$ is not contained in M_i for all $i \neq j$.

The structure of almost perfect rings is explained in the next three theorems, first dealing with <u>semiprime</u> rings, i.e., rings with zero nilradical.

THEOREM 1. A commutative semiprime Goldie ring is almost perfect if and only if it is a finite subdirect product of almost perfect domains.

THEOREM 2. A commutative ring R with ring of quotients Q is almost perfect if and only if:

(i) the nilradical N is T-nilpotent

(ii) the ring of quotients of R/N is isomorphic to Q/NQ

(iii) R/N is a (semiprime) almost perfect ring.

As a consequence, we have an intrinsic characterization, without mentioning the ring of fractions Q.

THEOREM 3. A commutative ring R is almost perfect if and only if:

(i) the nilradical N is T-nilpotent

(ii) a regular element of R/N has a regular representative in R

(iii) R/N is a finite subdirect product of almost perfect domains.

Theorem 3 has an important consequence in the Noetherian case.

COROLLARY. An almost perfect ring R is Noetherian if and only if it is a 1-dimensional Cohen-Macaulay ring.

6. Examples of almost perfect rings with zero-divisors

EXAMPLE 6.1. Let S be an APD and D a torsion-free divisible S-module.

Let R be the *idealization* of D, i.e., the ring of 2x2 matrices:

$$\begin{matrix} s & d \\ 0 & s \end{matrix}$$

with $s \in S$ and $d \in D$. Identify the matrix with the pair (s,d) and R with $S \oplus D$.

- $r = (s,d) \in R^x$ if and only if $s \neq 0$ (this depends on the torsion-freeness of D)
- if $r = (s,d) \in R^x$, then $Rr = (Ss,D)$ (this depends on the divisibility of D)
- if $r = (s,d) \in R^x$, then $R/Rr \cong S/Ss$, hence R is an almost perfect ring
- the ring of quotients Q of R is $Q = Q_0 \oplus D$, where Q_0 is the field of quotients of S, Q is a perfect ring with Jacobson radical $J = 0 \oplus D$ such that $J^2 = 0$.

EXAMPLE 6.2. Let S be a local APD with maximal ideal $P \neq 0$ and field of quotients Q_0. Let $R = \{ (s_1,s_2) \in S \times S \mid s_1 - s_2 \in P \}$.
R is a Goldie semiprime ring, hence an APR. The ring of quotients $Q = Q_0 \times Q_0$ is a perfect ring. R does not contain non-trivial idempotents, so R is not a direct product of APD's.

EXAMPLE 6.3. Let S be an APD with field of quotients Q. For each $n \geq 2$, the ring

$$R_n = (S + XQ[[X]]) / X^nQ$$

is an APR with nilradical $N_n = XQ[[X]]) / X^nQ$, nilpotent of degree n.
The ring $R^* = \prod_n R_n$ has nilradical $N = \prod_n N_n$, which fails to be T-nilpotent.
The subring R of R^* generated by N and the constant vectors (s, s, ...) has the factors R/Rr perfect for all $r \in R^x$, but its nilradical N is not T-nilpotent.

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