# On extremal product-one free sequences and weighted Davenport constants

Conference on Rings and Factorizations

Graz, Austria

**Sávio Ribas**

**Instituto Federal de Minas Gerais (IFMG) - Campus Ouro Preto**

Ouro Preto, Minas Gerais, Brazil

Joint work with **Fabio E. Brochero Martínez**/UFMG

February $19^{th}$-$23^{rd}$, 2018

## Theorem (Erdös-Ginzburg-Ziv, 1961)

*Let $n \in \mathbb{N}$. Given $a_1, a_2, \ldots, a_{2n-1} \in \mathbb{Z}$, there exist*

$$1 \leq i_1 < i_2 < \cdots < i_n \leq 2n-1$$

*such that*

$$a_{i_1} + a_{i_2} + \cdots + a_{i_n} \equiv 0 \pmod{n}.$$

The number $2n-1$ is the best possible, since the sequence

$$\underbrace{0, 0, \ldots, 0}_{n-1 \text{ times}}, \underbrace{1, 1, \ldots, 1}_{n-1 \text{ times}}$$

has no subsequence of length $n$ and sum $0 \in \mathbb{Z}_n$.

### Definition (small Davenport constant)

*Let $G$ a finite group (written multiplicatively). The* **small Davenport constant** *of $G$, $d(G)$, is the smallest number such that every sequence with $d(G)$ elements in $G$ (repetition allowed) contains some subsequence such that the product of its terms in some order is 1.*

- For every finite group $G$, we have $d(G) \leq |G|$.
- If $C_n$ is the cyclic group of order $n$ then $d(C_n) = n$.

- If $G = C_{n_1} \times \cdots \times C_{n_r}$ and $n_1 | \ldots | n_r$ then

$$d(G) \geq 1 + \sum_{i=1}^{r} (n_i - 1).$$

- Olson (1969) proved that the equality holds if each $n_i$ is a power of a prime or if $r = 2$.
- There are groups with $r \geq 4$ such that the inequality is strict.
- P. van Emde Boas & Kruyswijk (1969) proved that if $G$ is a finite abelian group then

$$d(G) \leq \exp(G) \left[1 + \log\left(\frac{|G|}{\exp(G)}\right)\right].$$

## Inverse zero-sum problems

Let $G$ a finite group written multiplicatively. By definition, there exist sequences $S = (x_1, x_2, \ldots, x_{d(G)-1})$ of $G$ such that

$$x_{i_1} \cdot x_{i_2} \ldots x_{i_k} \neq 1$$

for every non-empty subset $\{i_1, i_2, \ldots, i_k\} \subset \{1, 2, \ldots, d(G) - 1\}$.

The **inverse zero-sum problems** study the structure of these extremal sequences which are free of product-1 subsequences with some prescribed property.

The inverse problems associated to the Davenport constant were solved for few abelian groups. For example:

**Theorem (Inverse problem in $C_n$ wrt Davenport constant [6])**

*Let $n \geq 3$ and $S$ be a sequence in $C_n$ free of product-1 subsequences. Then there exist some $g \in S$ with multiplicity*

$$\geq 2|S| - n + 1.$$

*In particular,* $|S| = n - 1 \Rightarrow S = (\underbrace{g, \ldots, g}_{n-1 \text{ times}})$, *where $g$ generates $C_n$.*

## Metacyclic Groups

Let $q \in \mathbb{N}$, $m \geq 2$ be a divisor of $\varphi(q)$ and $s \in \mathbb{Z}_q^*$ such that $s^m \equiv 1 \pmod{q}$. Denote by $C_q \rtimes_s C_m$ the **Metacyclic Group**, i.e. the group generated by $x$ and $y$ with relations:

$$x^m = 1, \quad y^q = 1, \quad yx = xy^s.$$

Bass [2] showed that if $q$ is a prime and $ord_q(s) = m$ then

$$d(C_q \rtimes_s C_m) = m + q - 1.$$

## Theorem (Brochero Martínez, Ribas [3])

*Let $S$ be a sequence in the metacyclic group $C_q \rtimes_s C_m$, where $q$ is a prime and $\mathrm{ord}_q(s) = m$, with $|S| = m + q - 2$.*

1. *If $(m, q) \neq (2, 3)$ then the following statements are equivalent:*

    1.i *$S$ is free of product-1 subsequences;*

    1.ii *For some $1 \leq t \leq q-1$, $1 \leq i \leq m-1$ with $\gcd(i, m) = 1$ and $0 \leq \nu_1, \ldots, \nu_{m-1} \leq q-1$,*

$$S = (\underbrace{y^t, \ldots, y^t}_{q-1 \text{ times}}, x^i y^{\nu_1}, \ldots, x^i y^{\nu_{m-1}}).$$

2. *If $(m, q) = (2, 3)$ then $S$ is free of product-1 subsequences if and only if $S = (y^t, y^t, xy^\nu)$ for $t \in \{2, 3\}$ and $\nu \in \{0, 1, 2\}$ or $S = (x, xy, xy^2)$.*

## Definition ($A$-weighted Davenport constant)

*Let $G$ be an abelian group and $A \subset \mathbb{Z}$. The* **A-weighted Davenport constant** *of $G$, $D_A(G)$, is the smallest positive integer such that every sequence $x_1, \ldots, x_{D_A(G)}$ of $G$ has a non-empty subsequence $(x_{j_i})_i$ such that*

$$\sum_{i=1}^{t} \varepsilon_i x_{j_i} = 0$$

*for some $\varepsilon_i \in A$.*

- For every finite abelian group $G$, we have $D_A(G) \leq d(G)$.
- $D_A(G)$ is only known for some weight-sets $A$ and some "small" groups $G$. For example:

## Theorem (Adhikari, *et al* [1])

$$D_{\{\pm 1\}}(C_n) = \lfloor \log_2 n \rfloor + 1.$$

# Dihedral Groups

Let $n \geq 3$ be an integer. Denote by $D_{2n} \simeq (C_n \rtimes_{-1} C_2)$ the **Dihedral Group**, i.e. the group generated by $x$ and $y$ with relations:

$$x^2 = 1, \quad y^n = 1, \quad yx = xy^{-1}.$$

Zhuang and Gao [7] showed that $d(D_{2n}) = n + 1$.

## Theorem (Brochero Martínez, Ribas [4])

*Let $S$ be a sequence in the dihedral group $D_{2n}$ with $n$ elements.*

1. *If $n \geq 4$ then $S$ is free of product-1 subsequences if and only if*

$$S = (\underbrace{y^t, \dots, y^t}_{n-1 \text{ times}}, xy^s),$$

   *for some $1 \leq t \leq n-1$ with $\gcd(t, n) = 1$ and $0 \leq s \leq n-1$.*

2. *Case $n = 3$ is the same than case $(m, q) = (2, 3)$ of metacyclic groups.*

## Sketch of the proof

Let $H$ be the normal cyclic subgroup of order $n$ generated by $y$ and let

$$N = D_{2n} \setminus H = x \cdot H.$$

Define $k \in \mathbb{N}$ by the equation

$$|S \cap H| = n - k.$$

The case $k = 0$ implies we have $n$ elements in $H$, therefore there exist a product-1 subsequence.

The case $k = 1$ gives us the desired sequences.

Suppose $S \cap N = (xy^{\alpha_1}, \ldots, xy^{\alpha_k})$. We have

$$xy^{\alpha_i} \cdot xy^{\alpha_j} = y^{\alpha_j - \alpha_i}.$$

If $k \geq 2$ is "small" then almost every elements from $S \cap H$ are equal, and it is easy to exhibit a product-1 subsequence which involves pairs of elements from $S \cap N$.

If $k \geq 2$ is "large" then there exist a combination of a suitable subset of

$$\{\pm(\alpha_1 - \alpha_2), \pm(\alpha_3 - \alpha_4), \ldots, \pm(\alpha_{2\lfloor k/2 \rfloor - 1} - \alpha_{2\lfloor k/2 \rfloor})\}$$

summing 0 (by inverse Davenport constant of $C_n$), so we are done.

□

## Dicyclic Groups

Let $n \geq 2$ be an integer. Denote by $Q_{4n}$ the **Dicyclic Group**, i.e. the group generated by $x$ and $y$ with relations:

$$x^2 = y^n, \quad y^{2n} = 1, \quad yx = xy^{-1}.$$

Bass [2] showed that $d(Q_{4n}) = 2n + 1$.

If $n = 2$ then $Q_8$ is isomorphic to the well-known **Quaternion Group**:

$$\langle e, i, j, k | i^2 = j^2 = k^2 = ijk = e, e^2 = 1 \rangle.$$

The sequences with 4 elements in the quaternion group which are free of product-1 subsequences are $\pm(i, i, i, \pm j)$ and their respective simmetrics.

## Theorem (Brochero Martínez, Ribas [4])

*Let $S$ be a sequence in the dicyclic group $Q_{4n}$ with $2n$ elements.*

1. *If $n \geq 3$ then the following statements are equivalent:*
   - 1.i *$S$ is free of product-1 subsequences;*
   - 1.ii *For some $1 \leq t \leq n-1$ with $\gcd(t, 2n) = 1$ and $0 \leq s \leq 2n-1$,*

$$S = (\underbrace{y^t, \ldots, y^t}_{2n-1 \text{ times}}, xy^s).$$

2. *If $n = 2$ then $S$ is free of product-1 subsequences if and only if, for some $r \in \mathbb{Z}_4^*$ and $s \in \mathbb{Z}_4$, $S$ has one the following forms: $(y^r, y^r, y^r, xy^s)$, $(y^r, xy^s, xy^s, xy^s)$ or $(xy^s, xy^s, xy^s, xy^{r+s})$.*

This theorem is a corollary of the theorem for dihedral groups. The idea to prove it is using the equation $Q_{4n}/\{1, y^n\} \simeq D_{2n}$.

# The group $C_n \rtimes_s C_2$ with $s \not\equiv \pm 1 \pmod{n}$

## Theorem (Brochero Martínez, Ribas [5])

*Let $n \in \mathbb{N}$ such that there exist $s \in \mathbb{Z}_n^*$ satisfying $s^2 \equiv 1 \pmod{n}$ but $s \not\equiv \pm 1 \pmod{n}$. Also, let $n_1 = \gcd(s+1, n)$, $n_2 = n/n_1$ and assume that $\gcd(n_1, n_2) = 1$. Then*

$$D_{\{1,s\}}(C_n) \le \min\{n_2(\lfloor \log_2 n \rfloor + 1)\ ,\ \ 2n_2 + \lfloor n_1/2 \rfloor\}.$$

## Theorem (Brochero Martínez, Ribas [5])

*Let $n$, $s$, $n_1$ and $n_2$ as above. Suppose that $S$ is a sequence with $n$ elements in the group $G = C_n \rtimes_s C_2$ and assume $\min\{n_1, n_2\} \ge 20$. Then $S$ is free of product-1 subsequences if and only if*

$$S = (\underbrace{y^t, \dots, y^t}_{n-1 \text{ times}}, xy^r), \text{ for some } t \in \mathbb{Z}_n^* \text{ and } r \in \mathbb{Z}_n.$$

# References

[1] Adhikari, S.D., *et al*; **Contributions to zero-sum problems**. Discrete Math. 306 (2006), 1-10.

[2] Bass, J.; **Improving the Erdös-Ginzburg-Ziv theorem for some non-abelian groups**. J. Number Theory 126 (2007), 217-236.

[3] Brochero Martínez, F.E., Ribas, Sávio; **Extremal product-one free sequences in $C_q \rtimes_s C_m$**. Submitted. Available at: `arxiv.org/pdf/1610.09870.pdf` (2016).

[4] Brochero Martínez, F.E., Ribas, Sávio; **Extremal product-one free sequences in Dihedral and Dicyclic Groups**. Discrete Math. 341 (2018), 570-578.

[5] Brochero Martínez, F.E., Ribas, Sávio; **The $\{1, s\}$-weighted Davenport constant in $\mathbb{Z}_n$ and an application in an inverse problem**. Preprint (2018).

[6] Gao, W., Geroldinger, A., Schmid, W.A.; **Inverse zero-sum problems**. Acta Arith. 128.3 (2007), 245-279.

[7] Zhuang, J., Gao, W.; **Erdös-Ginzburg-Ziv theorem for dihedral groups of large prime index**. European J. Combin. 26 (2005), 1053-1059.

# Thank you!

Sávio Ribas

savio.ribas@gmail.com
savio.ribas@ifmg.edu.br