Factorizations of ideals in noncommutative rings similar to factorizations of ideals in commutative Dedekind domains

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Dedekind domains

Joseph Liouville, who was a member of the Académie des Sciences, immediately raised some doubts on Lamé's proof and, in particular, on the implicit assumption that the ring $\mathbb{Z}[\zeta_p]$ was a UFD for every prime integer p.

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With Kummer's work on the factorization theory in the case of cyclotomic integers, one began to pass from the study of factorization of elements to the study of factorization into prime ideals, (which might exist even if the element-wise factorization fails).

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With E. Noether (1927), S. Mori, K. Kubo, K. Matusita (about 1940), one arrives to the modern notion of Dedekind domain.

Dedekind domains

Theorem. The following conditions are equivalent for an integral domain R not a field:
(1) Every nonzero proper ideal of factors into prime ideals.
(2) R is Noetherian and its localizations at the maximal ideals are discrete valuation rings.
(3) Every nonzero fractional ideal of R is invertible.
(4) R is integrally closed, Noetherian, of Krull dimension one (i.e., every nonzero prime ideal is maximal).
(5) R is Noetherian, and for any two ideals I, J of R, $I \subseteq J$ if and only if there exists an ideal K of R such that $I = JK$.
Moreover, if these equivalent conditions hold, the factorization in (1) is necessarily unique up to the order of the factors.

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M_R is *uniserial* if its lattice of submodules is linearly ordered, that is, if for any submodules A, B of M_R either $A \subseteq B$ or $B \subseteq A$.

M_R is *serial* if it is a direct sum of uniserial sumodules.

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For every non-zero ideal I in a Dedekind domain R, the R-module R/I is serial, and this seems to be the motivation because of which Dedekind domains have such a good behavior as far as product decompositions of ideals is concerned. Thus we have studied the right ideals I in a (non-commutative) ring R for which the right R-module R/I is serial (i.e., a direct sum of finitely many uniserial right R-modules).

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1. *the same monogeny class*, denoted $[U]_m = [V]_m$, if there exist a monomorphism $U \to V$ and a monomorphism $V \to U$;
2. *the same epigeny class*, denoted $[U]_e = [V]_e$, if there exist an epimorphism $U \to V$ and an epimorphism $V \to U$.

Weak Krull-Schmidt Theorem

Theorem

[F., T.A.M.S. 1996] *Let* $U_1, \ldots, U_n, V_1, \ldots, V_t$ *be* $n+t$ *non-zero uniserial right modules over a ring* R. *Then the direct sums* $U_1 \oplus \cdots \oplus U_n$ *and* $V_1 \oplus \cdots \oplus V_t$ *are isomorphic* R*-modules if and only if* $n = t$ *and there exist two permutations* σ *and* τ *of* $\{1, 2, \ldots, n\}$ *such that* $[U_i]_m = [V_{\sigma(i)}]_m$ *and* $[U_i]_e = [V_{\tau(i)}]_e$ *for every* $i = 1, 2, \ldots, n$.

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Pavel Příhoda: an extension of the previous result to direct sums of infinite families of uniserial modules.

Coindependent submodules

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Coindependent submodules

Lemma

Let $A_1, \dots, A_n$ be proper right ideals of a ring R such that $A_iA_j = A_jA_i$ for every $i, j = 1, \dots, n$ and the family $\{A_1, \dots, A_n\}$ is coindependent. Then:

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(1) $A_1 \ldots A_n = \bigcap_{i=1}^{n} A_i$.

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(1) $A_1 \dots A_n = \bigcap_{i=1}^n A_i$.

(2) *If $n \geq 2$, then each A_i is a two-sided ideal.*

Serial factorizations

Definition. Let R be a ring. A *serial factorization* of a right ideal A of R is a factorization $A = A_1 \dots A_n$ with $\{A_1, \dots, A_n\}$ a coindependent family of proper right ideals of R, $A_iA_j = A_jA_i$ for every $i, j = 1, \dots, n$ and R/A_i a uniserial module for every $i = 1, \dots, n$.

Examples

Let R be a commutative PID. Then every non-zero ideal A of R has a serial factorization. If A is generated by a and $a = up_1^{t_1} \dots p_n^{t_n}$ is a factorization of a with u an invertible element and $p_1, \dots, p_n$ non-associate primes, then the serial factorization of A is $A = A_1 \dots A_n$ with $A_i = p_i^{t_i} R$. (This can be generalized to non-commutative right Bézout domains, that is, the integral domains in which every finitely generated right ideal is a principal right ideal.)

The example of Dedekind domains

More generally, let R be a commutative Dedekind domain, that is, an integral domain in which every non-zero ideal factors into a product of prime ideals. Then every non-zero ideal A of R has a serial factorization.

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More generally, let R be a commutative Dedekind domain, that is, an integral domain in which every non-zero ideal factors into a product of prime ideals. Then every non-zero ideal A of R has a serial factorization. Namely, let $A = P_1 \dots P_m$ be a factorization of A into a product of prime ideals P_i of R. Since R has Krull dimension one, the non-zero prime ideals P_i are maximal ideals of R.

The example of Dedekind domains

Without loss of generality, $A = P_1^{t_1} \dots P_n^{t_n}$ with $P_1, \dots, P_n$ distinct maximal ideals of R. It is easily seen that R/P^t is a uniserial module of finite composition length t for every integer $t \geq 0$ and every maximal ideal P of R.

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Right ideals with a serial factorization

Theorem. Let R be a ring, A a right ideal of R with a serial factorization $A = A_1 \dots A_n$ and B a right ideal of R containing A. Then:

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(1) B has a serial factorization if and only if either $B \supseteq A_i$ for some index $i = 1, \dots, n$ or B is a two-sided ideal of R.
(2) If B has a serial factorization, then the serial factorization of B is $B = (B + A_1) \dots (B + A_n)$ (where we are supposed to omit the factors $B + A_i$ equal to R).

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Theorem. Let R be a ring, and A, B be two similar right ideals of R. Suppose that A has a serial factorization. Then:
(1) B has a serial factorization.
(2) Either $A = B$ or the right R-module $R/A \cong R/B$ is uniserial.

Rigid factorizations

It is possible to generalize to the non-commutative setting the theory of semirigid GCD domains:

M. Zafrullah, *Semirigid GCD domains,* Manuscripta Math. **17** (1975), 55–66.

Right invariant elements

Let R be a (not necessarily commutative) integral domain. A non-zero element $a \in R$ is *right invariant* (P. M. Cohn) if $Ra \subseteq aR$.

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A non-zero element $a \in R$ is *right invariant* (P. M. Cohn) if $Ra \subseteq aR$. *Left invariant* elements are defined in a similar way, and an element a is *invariant* if it is left and right invariant, that is, if $Ra = aR \neq 0$.

Right invariant elements

The set $\mathrm{Inv}(R)$ of all invariant elements of an integral domain R is a multiplicatively closed subset of R that contains all invertible elements of R. Notice that, in an integral domain, an element is right invertible if and only if it is left invertible.

Rigid elements

An element a of an integral domain R is *rigid* if a is non-zero, non-invertible, and for every $x, y, x'y' \in R$, $a = xy' = yx'$ implies $x = yu$ or $y = xu$ for some $u \in R$.

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Two elements a, b of an integral domain R are *right associates* if there exists an invertible element $u \in R$ such that $a = bu$.

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(1) Each a_i is right invariant and rigid.
(2) The elements a_i and a_j are right coprime (that is, $a_iR + a_jR = R$) for every $i \neq j$,
(3) The elements a_ia_j and a_ja_i are right associates for every $i, j = 1, 2, \dots, n$.

We will call such a factorization $a = a_1 \dots a_n$ of a semirigid element $a \in R$ a *rigid factorization* of a.

Rigid factorizations and right Bézout domains

Theorem. Let R be a right Bézout domain and $a \in R$ be a semirigid element. Let $a = a_1 \dots a_n = b_1 \dots b_m$ be two rigid factorizations of A. Then $n = m$ and there exists a unique permutation σ of $\{1, \dots, n\}$ such that a_i and $b_{\sigma(i)}$ are right associates for every $i = 1, \dots, n$.

Describing factorizations

A. Facchini and Z. Nazemian, *Serial factorizations of right ideals*, accepted for publication in J. Pure Appl. Algebra, 2018, available in http://arxiv.org/abs/1802.03786

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A. Facchini and M. Fassina, *Factorization of elements in noncommutative rings, II*, Comm. Algebra, published online (2017)

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What is the algebraic object that describes the factorizations of an element in a ring (possibly non-commutative, possibly with zero-divisors)?

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What is the algebraic object that describes the factorizations of an element in a ring (possibly non-commutative, possibly with zero-divisors)?

It is the set of (ascending) chains in a partially ordered set. (This is the analogue of the fact that to describe finite direct-sum decompositions of a module the convenient algebraic structure is a commutative monoid, possibly with order-unit). The idea is essentially taken from P. M. Cohn, *Unique factorization domains*, Amer. Math. Monthly **80** (1973), 1–18.

Describing factorizations

For any ring R with identity, the modular lattice $\mathcal{L}(R_R)$ of all right ideals of R has as a subset the set $\mathcal{L}_p(R_R) := \{ aR \mid a \in R \}$ of all principal right ideals of R. The lattice structure on $\mathcal{L}(R_R)$ induces a partial order on $\mathcal{L}_p(R_R)$.

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Describing factorizations

Theorem Let a be an element of a ring R,

$$\mathcal{F}(a) := \{\, (a_1, a_2, \dots, a_n) \mid n \geq 1,\ a_i \in R,\ a_1 a_2 \dots a_n = a \,\}$$

the set of all factorizations of a, and

$$\mathcal{C}_a := \{\, (aR, I_1, I_2, \dots, I_{n-1}, R) \mid n \geq 1, I_j \text{ a principal right ideal of } R \,\}$$

the set of all finite chains of principal right ideals from aR to R_R.
Let $f \colon \mathcal{F}(a) \to \mathcal{C}_a$ be the mapping defined by

$$f(a_1, a_2, \dots, a_n) = (aR = a_1 a_2 \dots a_n R, a_1 a_2 \dots a_{n-1} R, \dots, a_1 R, R)$$

for every $(a_1, a_2, \dots, a_n) \in \mathcal{F}(a)$. Then the mapping f is surjective, and two factorizations in $\mathcal{F}(a)$ are mapped via f to the same element of $\mathcal{C}_a$ if and only if they are equivalent factorizations of a.

Describing factorizations

Here two factorizations $(a_1, a_2, \dots, a_n), (b_1, b_2, \dots, b_m)$ of an element $a \in R$ are *equivalent* if $n = m$ and there exist $u_1, v_1, u_2, v_2, \dots, u_{n-1}, v_{n-1} \in R$ and $t_i \in \mathrm{r.ann}(a_1 a_2 \dots a_{i-1} u_{i-1})$ for every $i = 1, 2, \dots, n$ such that $u_i v_i - 1 \in \mathrm{r.ann}_R(a_1 a_2 \dots a_i)$ for every $i = 1, 2, \dots, n-1$ and $(b_1, b_2, \dots, b_m) = (a_1 u_1, v_1 a_2 u_2 + t_2, v_2 a_3 u_3 + t_3, \dots, v_{n-1} a_n + t_n)$.

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$(b_1, b_2, \ldots, b_m) = (a_1 u_1, v_1 a_2 u_2 + t_2, v_2 a_3 u_3 + t_3, \ldots, v_{n-1} a_n + t_n)$.

(Two factorizations $(a_1, a_2, \ldots, a_n), (b_1, b_2, \ldots, b_m)$ into right regular elements are equivalent if and only if $n = m$ and there exist $u_1, u_2, \ldots, u_{n-1} \in U(R)$ such that
$(b_1, b_2, \ldots, b_m) = (a_1 u_1, u_1^{-1} a_2 u_2, u_2^{-1} a_3 u_3, \ldots, u_{n-1}^{-1} a_n)$.)

Describing factorizations

Nice example concerning right noetherian right chain ring and factorizations of matrices.