Geometric and topological applications to intersections of valuation rings

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Framework

Problem: Find a framework for classifying/describing/studying integrally closed domains when viewed **as intersections of valuation rings**.

In this talk, we outline one possible framework.

Basic idea:

(a) The space of valuation rings is a locally ringed space.

(b) Intersections of valuation rings are rings of sections.

(c) Features can be distinguished by morphisms into the projective line.

Statements (a) and (b) are trivial observations.

Statement (c) is the claim that the talk hopes to justify.

The Zariski-Riemann space

Let F be a field and D be a subring of F (e.g., D is prime subring of F).

Zariski-Riemann set: $\mathfrak{X} =$ the set of all valuation rings of F containing D.

Zariski-Riemann subset: $\mathfrak{X}_R = \{V \in \mathfrak{X} : R \subseteq V\}$ when $R \subseteq F$

Finitely generated Zariski-Riemann subset if R is a f.g. D-algebra.

Definition. The Zariski-Riemann space of F/D is the Zariski-Riemann set with open basis the **finitely generated** Zariski-Riemann subsets.

Source of many woes: Union of ZR subspaces may not be a ZR subspace.

...But it is true for Prüfer domains.

Why the name?

Nagata, 1962:

> *The name of Riemann is added because Zariski called this space 'Riemann manifold' in the case of a projective variety, though this is not a Riemann manifold in the usual sense in differential geometry. The writer believes that the motivation for the terminology came from the case of a curve. Anyway, the notion has nearly nothing to do with Riemann, hence the name 'Zariski space' is seemingly preferable. But, unfortunately, the term 'Zariski space' has been used in a different meaning [e.g., a Noetherian topological space for which every nonempty closed irreducible subset has a unique generic point]. Therefore we are proposing the name 'Zariski-Riemann space'.*

Topological features

Theorem. (Zariski, 1944)

Zariski-Riemann spaces are quasicompact.

Why Zariski cared: finite resolving system can replace an infinite one.

Theorem (Dobbs-Fontana; Heubo-Kwegna; Fontana-Finocchiaro-Loper)

Zariski-Riemann space is spectral.

Proof: $\mathfrak{X} \simeq \mathrm{Spec}(\mathrm{Kr}(F/D))$ via Halter-Koch's notion of F-function rings.

Aside: The prime spectrum of the Kronecker function ring can be thought of as a "ruled" version of $\mathfrak{X}$; the ruling makes the locally ringed space $\mathfrak{X}$ into an affine scheme that encodes the valuation theory of F/D.

Zariski vs. inverse topology

$$\{\text{ZR subspaces}\} \not\subseteq \{\text{Zariski open subsets}\} \cup \{\text{Zariski closed subsets}\}$$

$$\{\text{finite unions of f.g. ZR subspaces}\} = \{\text{Zariski quasicompact open sets}\}$$

(can omit "finite unions of" for Prüfer domains!)

Inverse topology: Use **quasicompact open sets** as basis of **closed** sets.

$$\text{inverse closed set} = \bigcap (\text{finite union of f.g. ZR spaces})$$

$$\{\text{ZR subspaces}\} \subseteq \{\text{inverse closed subsets}\}$$

(equality for Prüfer domains!)

Representing subspaces

A subspace Z of $\mathfrak{X}$ represents a ring R if $R = \bigcap_{V \in Z} V$.

Krull: Every integrally closed domain R can be thus represented.

Clumsiest choice: $\mathfrak{X}_R$ represents R.

Crucial point: Many inverse closed subsets of $\mathfrak{X}_R$ can better represent R.

...They correspond precisely to the Kronecker function rings of R.

So if R is Prüfer, $\mathfrak{X}_R$ is the **only** inverse closed subset that represents R.

Inverse topology is subtle enough to help with representations...

Topological approach

A crucial test case for any framework for studying intersections of valuation rings is whether the framework can detect Prüfer domains.

Topological approach fails miserably:

> *Every Zariski-Riemann space is homeomorphic (in Zariski, inverse or patch topology) to Zariski-Riemann space of a Prüfer domain.*

Another test case: Can the framework detect irredundant members of a representing set of valuation rings?

This fails too, but less miserably:

> *A valuation ring that is an* **isolated point** *in a representing set with respect to the inverse or patch topology is irredundant. But this property is not necessary.*

Geometrical approach

For each nonempty subset U of $\mathfrak{X}$, let $\mathcal{O}(U) = \bigcap_{V \in U} V$.

$\mathcal{O}$ is a sheaf of rings with stalks the valuations rings in $\mathfrak{X}$

$\Rightarrow \quad (\mathfrak{X}, \mathcal{O})$ is a locally ringed space.

Same idea works for any irreducible subspace of $\mathfrak{X}$.

...Irreducibility needed for sheaf axiom (enough to contain F).

Each irreducible subspace Z of $\mathfrak{X}$ is thus a locally ringed space.

Integrally closed rings thus occur as rings of sections of appropriate sheaf.

...But in itself this idea is too inert to say anything new!

Structure of $(\mathfrak{X}, \mathcal{O})$

In general, the locally ringed space $(\mathfrak{X}, \mathcal{O})$ is not a scheme (but it is a projective limit of projective schemes.)

affine scheme $= \mathrm{Spec}(R)$ equipped with Zariski topology and a sheaf of rings that varies continuously over the space and has stalks R_P.

scheme $=$ locally ringed space having an open cover of affine schemes.

Proposition.
$\mathfrak{X}$ is a scheme **iff**

(i) each valuation ring in $\mathfrak{X}$ is a localization of one of the R_i, and

(ii) each R_i is a Prüfer domain with quotient field F.

E.g., $\mathfrak{X}$ is an affine scheme **iff** D is a Prüfer domain with quotient field F.

Morphisms into projective space

Proposition.
$Z \subseteq \mathfrak{X}$ is an affine scheme **iff**
Z is inverse closed and $A = \bigcap_{V \in Z} V$ is a Prüfer domain with q.f. F.

So to detect when an intersection of valuation rings is Prüfer is the same as detecting when a subspace of $\mathfrak{X}$ is an affine scheme.

We do this through morphisms into the projective line.

$$\begin{aligned}\mathbb{P}^1_D &= \mathrm{Proj}(D[T_0, T_1]) = \text{projective line over } D \\ &= \text{homogeneous primes in } \mathrm{Spec}(D[T_0, T_1]) \text{ not containing } (T_0, T_1).\end{aligned}$$

A morphism $Z \to \mathbb{P}^1_D$ consists of:

- continuous map: $Z \xrightarrow{f} \mathbb{P}^1_D$
- sheaf morphism: $f_* \mathcal{O}_Z \xleftarrow{f^\#} \mathcal{O}_{\mathbb{P}^1_D}$ (assembles ring homomorphisms)

Theorem.

$A = \bigcap_{V \in Z} V$ is Prüfer with quotient field F	$\iff$	every D-morphism $Z \to \mathbb{P}^1_D$ factors through an affine scheme.

Theorem.

$A = \bigcap_{V \in Z} V$ is Prüfer with quotient field F and torsion Picard group	$\Longleftarrow$	image of each D-morphism $Z \to \mathbb{P}^1_D$ is in a distinguished affine open subset of $\mathbb{P}^1_D$

When $f \in D[T_0, T_1]$ is homogeneous of positive degree, then

$$(\mathbb{P}^1_D)_f = \{P \in \mathbb{P}^1_D : f \notin P\}$$

is a distinguished affine open subset of $\mathbb{P}^1_D$.

So $\mathbb{P}^1_D$ is covered by many affine open subsets.

What conditions guarantee $Z \to \mathbb{P}^1_D$ lands in one of them?

Applications

Three classical independent results about Prüfer intersections can now be reduced to prime avoidance arguments...

Corollary. (Nagata) $A = \bigcap_{V \in Z} V$ is Prüfer when Z is finite.

Proof.

Let $\phi : Z \to \mathbb{P}^1_D$ be a morphism.

Its image is finite.

Prime Avoidance $\Rightarrow \exists f \in D[T_0, T_1]$ not in any prime ideal in Im ϕ.

$\{P \in \mathbb{P}^1_D : f \notin P\}$ is an **affine** open set containing Im ϕ.

So by the theorem, A is Prüfer.

(In fact, f can be chosen to be linear and this implies that A is Bézout.)

Corollary. (Dress, Gilmer, Loper, Roquette, Rush)
$A = \bigcap_{V \in Z} V$ is Prüfer when there exists a nonconstant monic polynomial $f \in A[T]$ which has no root in residue field of any $V \in Z$.

Proof.

Let $\phi : Z \to \mathbb{P}^1_D$ be a morphism.

Let $\overline{f}$ be the homogenization of f.

Then $\{P \in \mathbb{P}^1_D : \overline{f} \notin P\}$ is an **affine** open set containing Im ϕ.

So by the theorem, A is Prüfer with torsion Picard group.

Corollary. (Roitman)

$A = \bigcap_{V \in Z} V$ is Bézout when A contains a field of cardinality $> |Z|$.

Proof.

Let $\phi : Z \to \mathbb{P}^1_D$ be a morphism.

Use the fact that there are more units in A than valuation rings in Z to construct a homogeneous $f \in D[T_0, T_1]$ that is not contained in any prime ideal in the image of ϕ.

Then $\{P \in \mathbb{P}^1_D : f \notin P\}$ is an **affine** open set containing Im ϕ.

So by the theorem, A is Prüfer.

(In fact, f can be chosen to be linear, so A is a Bézout domain.)

Special Case

Assume:

- D is a local subring of F (e.g., D is a subfield)
- all but at most finitely many valuation rings in Z dominate D.

Theorem.

$A = \bigcap_{V \in Z} V$ is Prüfer with quotient field F and torsion Picard group $\Longleftarrow$ no D-morphism $Z \to \mathbb{P}^1_D$ has every closed point in its image.

So if Z maps onto $\mathbb{P}^1_D$, then $A = \bigcap_{V \in Z} V$ is **not** a Prüfer domain.

Corollary.

If $|Z| < |D/m|$, then $A = \bigcap_{V \in Z} V$ is a Bézout domain.

Local uniformization

Possibly vacuous application...

Suppose that D has quotient field F.

A valuation ring V in $\mathfrak{X}$ admits local uniformization if $\exists$ projective model X of F/D such that V dominates a regular local ring in X.

Longstanding problem in Resolution of Singularities: Does local uniformization hold in positive characteristic with dimension > 3?

Corollary

$D =$ quasi-excellent local Noetherian domain with quotient field F.

$Z =$ valuation rings that dominate D but don't admit local uniformization.

Then $\bigcap_{V \in Z} V$ is a Prüfer domain with torsion Picard group.

No claim that this contributes to the longstanding problem! The corollary is only a curiosity...

Irredundance

V is irredundant in Z if $\displaystyle\bigcap_{U \in Z} U \subsetneq \bigcap_{U \in Z \setminus \{V\}} U.$

Nice fact: V irredundant in Z and V has rational value group
$\Rightarrow V$ is a localization of $\mathcal{O}(Z) = \bigcap_{V \in Z} V$.

Theorem.
V is irredundant in $Z \iff \exists$ morphism $Z \to \mathbb{P}^1_D$ that distinguishes between the images of V and $Z \setminus \{V\}$.

"Distinguishes" means there exists a D-morphism $(f, f^{\#}) : Z \to \mathbb{P}^1_D$ and an open affine subset of $\mathbb{P}^1_D$ that contains $f(Z \setminus \{V\})$ but not $f(V)$.

Recall that topology alone can't detect irredundance.

Overrings of two-dimensional Noetherian domains

Rest of talk: D is a two-dimensional Noetherian domain with q.f. F.

Goal: Describe the integrally closed rings $\bigcap_{V \in Z} V$ between D and F.

Special case: $\exists$ morphism $Z \to \mathbb{P}^1_D$ with "small" fibers.

This is in keeping with the philosophy of understanding intersections of valuation rings when there are not "too many" of them.

Theorem.

$\exists$ morphism $Z \to \mathbb{P}^1_D$ with **Noetherian** fibers

$\Rightarrow \exists$ unique strongly irredundant representation of $\mathcal{O}(Z) = \bigcap_{V \in Z} V$.

Theorem. (Local classification)

If there exists a morphism $Z \to \mathbb{P}^1_D$ with **Noetherian fibers**, then for each prime ideal P of $A = \mathcal{O}(Z)$, $\exists$ integrally closed Noetherian overring B of A_P such that one of the following holds:

(a) A_P is a valuation ring;

(b) A_P is a Noetherian ring;

(c) $\exists!$ irrational valuation rings $V_1, \ldots, V_n$ with $A_P = V_1 \cap \cdots \cap V_n \cap B$, and each V_i irredundant;

(d) $\exists!$ irrational valuation rings $V_1, \ldots, V_n$ and a unique collection Γ of valuation overrings of A of Krull dimension 2 such that $A_P = V_1 \cap \cdots \cap V_n \cap (\bigcap_{V \in \Gamma} V) \cap B$, and each V_i, V is strongly irredundant in this intersection; or,

(e) $\exists!$ collection Γ of valuation overrings of A of Krull dimension 2, all in Z, such that $A_P = (\bigcap_{V \in \Gamma} V) \cap B$, and each V is strongly irredundant in this intersection.

Theorem. (Technical, but hopefully a prototype for strong results)
Suppose:

- the image of Z in $\mathrm{Spec}(D)$ is not dense,
- some finite Cantor-Bendixson patch derivative is empty, and
- Z consists of DVRs.

Then $A = \bigcap_{V \in Z} V$ is a Bézout almost Dedekind domain.

The second condition is satisfied if the process

> *...set of limit points of (set of limit points of(...set of limit points of (set of limit points of Z))))*

reaches the empty set in finitely many steps.

The theorem can likely be viewed in terms of strong approximation...

Thank you