One-dimensional stable domains

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Motivations

Stability of ideals arose in the sixties of last century in the context of one-dimensional Noetherian rings, in particular in the study of reflexive rings and decomposition of torsion free modules made by H. Bass [*On the ubiquity of Gorenstein rings*, Math Z. (1963)] and E. Matlis [*Reflexive domains*, J. Algebra (1968)].

This notion was explicitly introduced by J. Lipman in order to study Arf rings [*Stable ideals and Arf rings*, Amer. J. Math (1971)] and deepened by J.D. Sally and W.V. Vasconcelos [*Stable rings and a problem of Bass*, Bull. AMS (1973), *Stable rings*, J. Pure Appl. Algebra (1974)] and D.E. Rush [*Rings with two-generated ideals* J. Pure Appl. Algebra (1991), *Two-generated ideals and representations of abelian groups over valuation rings*, J. Algebra (1995)].

We are interested in stability of domains.

The notion of stability for arbitrary integral domains was firstly considered by D.D. Anderson, J. Huckaba and I. Papick [*A note on stable domains*, Houston J. Math (1987)].

Since 1998, stability of ideals has been thoroughly investigated by B. Olberding in several papers: [*Globalizing local properties of Prüfer domains*, J. Algebra (1998), *On the classification of stable domains*, J. Algebra (2001), *On the structure of stable domains*, Comm. Algebra (2002), *Noetherian rings without finite normalizations*, Progress in commutative algebra 2, Walter de Gruyter (2012), *One-dimensional bad Noetherian domains*, Trans. AMS (2014), *Finitely Stable Rings* (2014)].

R will always denote a domain. For brevity, by ideal we mean a nonzero fractional ideal of R.

An ideal I of a domain R is called *stable* if it is invertible in its endomorphism ring $E(I) := (I : I)$. Invertible ideals are clearly stable.

A domain R is called *stable* if each ideal is stable and *finitely stable* if each finitely generated ideal is stable.

Of course in the Noetherian case stability and finite stability coincide.

Olberding showed that the study of stability can be reduced to the local case:

R is finitely stable $\Leftrightarrow$ R_M is finitely stable, for each maximal ideal M.

R is stable $\Leftrightarrow$ R_M is stable, for each maximal ideal M, and R has finite character.

In the classification of stable domains, the integrally closed case and one-dimensional case are particularly interesting.

If R is integrally closed and I is a finitely generated ideal, we have $R = (I : I)$. So that a finitely generated stable ideal I is invertible. Hence, a local finitely stable domain is a valuation domain and globalizing:

R integrally closed and finitely stable $\Leftrightarrow$ R Prüfer.

However, a stable valuation domain has to be strongly discrete. So that

R integrally closed and stable $\Leftrightarrow$ R strongly discrete Prüfer with finite character.

Recall that a Prüfer domain R is *strongly discrete* if PR_P is principal, for each prime ideal P.

By the classification given by Olberding, a local domain R is stable if and only if it arises from a pullback diagram of type:

$$\begin{array}{ccc} R & \longrightarrow & D \\ \downarrow & & \downarrow \\ V & \xrightarrow{\varphi} & \frac{V}{I} \cong T(D) \end{array}$$

where V is a strongly discrete valuation domain, I is an ideal of V, D is a local stable ring of dimension at most one with some conditions on the zerodivisors, and V/I is isomorphic to the total quotient ring $T(D)$ of D.

In particular, D can be a one-dimensional stable local domain.

One-dimensional stable domains

It is known that:

- Stable Noetherian domains are one-dimensional.

The converse is true if R has nonzero conductor in its integral closure R'.

- If R is stable one-dimensional and $(R : R') \neq (0)$, R is Noetherian 2-generated.

On the other hand,

- If R is local stable and $(R : R') = (0)$, R is one-dimensional and R' is a one-dimensional discrete valuation ring (DVR).

Olberding constructed several examples of local stable domains such that $(R : R') = (0)$. These domains can be chosen to be Noetherian, Noetherian 2-generated or not Noetherian.

So that a one-dimensional stable domain need not be Noetherian.

We proved that a stable one-dimensional domain is Mori and is precisely a Mori finitely stable domain. In particular, stability and finite stability coincide for Mori domains.

Recall that the *divisorial closure* of an ideal I is the ideal $I_v := (R : (R : I))$ and that I is called *divisorial* if $I = I_v$.

A *Mori domain* is a domain satisfying the ascending chain condition on divisorial ideals. Clearly Noetherian domains are Mori.

The Theorem

The following conditions are equivalent for a domain R (not necessarily local):

(i) R is stable and one-dimensional;

(ii) R is Mori and stable;

(iii) R is Mori and finitely stable.

In addition, under the previous conditions, each nonzero ideal I of R is 2-v-generated: that is $I_v = \langle x, y \rangle_v$, for some $x, y \in I$ (equivalently $(R : I) = (R : \langle x, y \rangle)$).

• All the examples constructed by Olberding of local one-dimensional stable domains which are not Noetherian are new examples of Mori domains.

• A local one-dimensional stable domain R cannot arise as a proper pullback like before. In fact, in a pullback of that type

$$\begin{array}{ccc} R & \longrightarrow & D \\ \downarrow & & \downarrow \\ V & \xrightarrow{\varphi} & \frac{V}{I} \cong T(D) \end{array}$$

R is Mori if and only if V is a DVR and D is a field (Mimouni, 1997).

Sketch of the proof

We can assume that $R := (R, M)$ is local.

In this case, for the proof of the theorem it is relevant the Archimedean property of Mori domains.

A domain R is called *Archimedean* if $\bigcap_{n\geq 1} a^n R = (0)$, for each nonunit $a \in R$. One-dimensional and accp domains are Archimedean.

It was already known that a Mori stable domain is one-dimensional (Kabbaj-Mimouni, 2003). We prove that in the local case all the stable Archimedean domains are one-dimensional.

For R local stable, we prove:

(i) R one-dimensional $\Rightarrow$ (ii) R Mori 2-v-generated $\Rightarrow$ R Archimedean $\Rightarrow$ (i) R one-dimensional.

In addition:

(iii) R finitely stable Mori $\Rightarrow$ (ii) R stable.

We use the structure of the integral closure of a local stable domain given by Olberding.

Since stability passes to overrings, the integral closure R' of a stable domain R is a strongly discrete Prüfer domain with finite character.

If $R := (R, M)$ is a stable local domain, R' can be constructed as a union of an ascending chain of overrings.

Let $R := (R, M)$ be a local domain and consider the chain of overrings:

$$\begin{aligned} R \subseteq R_1 \subseteq R_2 \subseteq \dots \\ \subseteq R_n \subseteq \dots \subseteq T := \bigcup_{n \geq 0} R_n. \end{aligned}$$

Where:

- $R_1 := E(M) := (M : M)$,

and, for $n \geq 1$,

- $R_{n+1} = E(M_n) = (M_n : M_n)$ if R_n is local with maximal ideal denoted by M_n, and

- $R_{n+1} = R_n$ if R_n is not local.

If $R := (R, M)$ is stable, $R' = T := \bigcup_{n \geq 0} R_n$. More generally, if R is finitely stable with stable maximal ideal, $T \subseteq R'$.

Thus, if $R := (R, M)$ is stable, we have two cases:

- R_k is local for $k \leq n-1$ and R_n is not local. Then:

$$\begin{aligned} R \subseteq R_1 := (M : M) \subseteq R_2 := (M_1 : M_1) \subseteq \dots \\ \subseteq R_{n-1} := (M_{n-2} : M_{n-2}) \subseteq R' = T := R_n. \end{aligned}$$

In this case, $R' = R_n$ is a strongly discrete Prüfer domain with two maximal ideals and R' is a finitely generated R-module. Thus $(R : R') \neq (0)$.

- R_n is local for $n \geq 1$. Then:

$$R \subseteq R_1 := (M : M) \subseteq R_2 := (M_1 : M_1) \subseteq \dots$$
$$\subseteq R_n := (M_{n-1} : M_{n-1}) \subseteq \dots \subseteq R' = T := \bigcup_{n \geq 0} R_n.$$

In this case, $R' = \bigcup_{n \geq 0} R_n$ is a strongly discrete valuation domain with maximal ideal

$$mR' = MR' = \bigcup_{n \geq 0} M_n,\ m \in M.$$

Furthermore,

$$M_n = m(M_n : M_n) = mR_{n+1} = MR_{n+1}, \text{ for all } n \geq 0.$$

Let $R := (R, M)$ local.

(i) R stable one-dimensional $\Rightarrow$ $(R : I) = (R : \langle x, y \rangle)$, $x, y \in I$ $\Rightarrow$ (ii) R Mori.

- $(R : R') \neq (0)$ $\Rightarrow$ R is Noetherian 2-generated.

- $(R : R') = (0)$ $\Rightarrow$ R_n is local for each $n \geq 1$ and $R' = T := \bigcup_{n \geq 0} R_n$ is a DVR with maximal ideal mT.

Let I be an ideal of R. Multiplying if necessary by some element of $Qf(R)$ we can assume that $(R : I) \subseteq T = R'$. Let v be the normalized valuation on R'. There exists $x \in I$ of minimal value $k := v(x)$ such that $x/m \notin R$. Also, there exists a unit u of R_k and $y \in I$ such that $y/um \notin R$. Then $(R : I) = (R : \langle x, y \rangle)$.

(R, M) stable Archimedean $\Rightarrow$ (R, M) (stable) one-dimensional.

If R is Archimedean, the overring $(I : I)$ is Archimedean, for each nonzero ideal I. Hence, by induction, each overring $R_n := (M_{n-1} : M_{n-1})$ is Archimedean.

Since $R' = T := \bigcup_{n \geq 0} R_n$ has principal maximal ideals, this implies that R', and so R, is one-dimensional.

A modification of the proof shows that more generally:

(R, M) Archimedean finitely stable with stable maximal ideal $\Rightarrow$ (R, M) one-dimensional.

(iii) (R, M) finitely stable Mori $\Rightarrow$ (ii) (R, M) stable (Mori).

We use the fact that a local one-dimensional finitely stable domain is stable if and only if its maximal ideal is stable (Olberding).

Indeed,

The maximal ideal of a local finitely stable Mori domain is divisorial and stable. Since a local Mori finitely stable domain with stable maximal ideal is one-dimensional, it follows that it is stable.

Summarizing, the following conditions are equivalent for a local domain R:

(i) R is stable and one-dimensional;

(ii) R is Mori and stable;

(iii) R is Mori and finitely stable;

(iv) R is Archimedean and stable;

(v) R is Archimedean, finitely stable with stable maximal ideal.

However examples show that a stable Archimedean domain that is not semilocal need not be one-dimensional (equivalently Mori).

A conjecture

We have seen that:

R Mori stable $\Rightarrow$ R Mori 2-v-generated.

The converse is known to be true if each ideal of R is divisorial (i.e: R is a *divisorial domain*). A divisorial Mori domain is Noetherian.

R Noetherian 2-generated domain $\Leftrightarrow$ R Noetherian stable and divisorial.

However, we cannot discard the hypothesis of divisoriality:

R Mori 2-v-generated domain $\neq>$ R stable

(take a Krull domain that is not Dedekind).

Conjecture: A Mori 2-v-generated domain is v-stable

An ideal I is *v-stable* if it is v-invertible in the overring $(I_v : I_v)$ of R and R is v-stable if each ideal is v-stable.

Clearly stable implies v-stable.