Franz Halter-Koch's contributions to ideal systems: a survey of some selected topics

Marco Fontana

Dipartimento di Matematica e Fisica
Università degli Studi "Roma Tre"

Graz, September 2014

§0. Introduction

What is now called *Multiplicative Ideal Theory* has its origin in R. Dedekind's work published in 1871 and was later developed in a more general context by W. Krull, E. Noether and H. Prüfer about 1930.

P. Lorenzen in 1939 was probably the first to take a new point of view: investigate the multiplicative structure without making reference, as far as possible, to the additive structure. He presented an axiomatic treatment of the theory of ideal systems in monoids and groups generalizing parts of the results obtained by Dedekind and Krull.

With a similar point of view, P. Jaffard in 1960 in his book *"Les Systèmes d'Idéaux"* provided a systematic study of the multiplicative theory of ideal systems. However, his original style, not easy to read, greatly limited the diffusion of his work and several of his results were rediscovered later by various authors.

§0. Introduction

What is now called *Multiplicative Ideal Theory* has its origin in R. Dedekind's work published in 1871 and was later developed in a more general context by W. Krull, E. Noether and H. Prüfer about 1930.

P. Lorenzen in 1939 was probably the first to take a new point of view: investigate the multiplicative structure without making reference, as far as possible, to the additive structure. He presented an axiomatic treatment of the theory of ideal systems in monoids and groups generalizing parts of the results obtained by Dedekind and Krull.

With a similar point of view, P. Jaffard in 1960 in his book *"Les Systèmes d'Idéaux"* provided a systematic study of the multiplicative theory of ideal systems. However, his original style, not easy to read, greatly limited the diffusion of his work and several of his results were rediscovered later by various authors.

§0. Introduction

What is now called *Multiplicative Ideal Theory* has its origin in R. Dedekind's work published in 1871 and was later developed in a more general context by W. Krull, E. Noether and H. Prüfer about 1930.

P. Lorenzen in 1939 was probably the first to take a new point of view: investigate the multiplicative structure without making reference, as far as possible, to the additive structure. He presented an axiomatic treatment of the theory of ideal systems in monoids and groups generalizing parts of the results obtained by Dedekind and Krull.

With a similar point of view, P. Jaffard in 1960 in his book *"Les Systèmes d'Idéaux"* provided a systematic study of the multiplicative theory of ideal systems. However, his original style, not easy to read, greatly limited the diffusion of his work and several of his results were rediscovered later by various authors.

Important improvements and generalizations to Lorenzen's theory were due to Karl Egil Aubert starting in 1953.

In the 70ties, the books *"Multiplicative Theory of Ideals"* by Larsen-McCarty published in 1971 and *"Multiplicative Ideal Theory"* by R. Gilmer (1968 & 1972) provide a more modern and systematic approach to Dedekind, Kronecker, Krull, Prüfer classical multiplicative ideal theory in the context of integral domains.

Finally in 1998 Franz Halter-Koch published his *"Ideal Systems: An introduction to multiplicative ideal theory"* which is considered a fundamental treatise on these topics, in the very general language of ideal systems on commutative monoids.

Important improvements and generalizations to Lorenzen's theory were due to Karl Egil Aubert starting in 1953.

In the 70ties, the books *"Multiplicative Theory of Ideals"* by Larsen-McCarty published in 1971 and *"Multiplicative Ideal Theory"* by R. Gilmer (1968 & 1972) provide a more modern and systematic approach to Dedekind, Kronecker, Krull, Prüfer classical multiplicative ideal theory in the context of integral domains.

Finally in 1998 Franz Halter-Koch published his *"Ideal Systems: An introduction to multiplicative ideal theory"* which is considered a fundamental treatise on these topics, in the very general language of ideal systems on commutative monoids.

Important improvements and generalizations to Lorenzen's theory were due to Karl Egil Aubert starting in 1953.

In the 70ties, the books *"Multiplicative Theory of Ideals"* by Larsen-McCarty published in 1971 and *"Multiplicative Ideal Theory"* by R. Gilmer (1968 & 1972) provide a more modern and systematic approach to Dedekind, Kronecker, Krull, Prüfer classical multiplicative ideal theory in the context of integral domains.

Finally in 1998 Franz Halter-Koch published his *"Ideal Systems: An introduction to multiplicative ideal theory"* which is considered a fundamental treatise on these topics, in the very general language of ideal systems on commutative monoids.

§1. Notation and Basic Definitions

An hereditary torsion theory for a commutative ring R is characterized by the family $\mathcal{F}$ of the ideals I of R for which R/I is a torsion module (for more details cf. B. Stenström's book "Rings of Quotients", Springer, Berlin 1975; Ch. VI).

It turns out that such a family $\mathcal{F}$ of ideals is the family of the neighborhoods of 0 for a certain linear topology of R.

The notion of localizing system (or topologizing system) was introduced (in a more general context) by P. Gabriel in order to characterize such topologies from an ideal-theoretic point of view (cf. Pierre Gabriel, La localisation dans les anneaux non commutatifs Exposé No. 2, in Séminaire Dubreil, Algèbre et théorie des nombres, 1959-1960; N. Bourbaki, 1961, Ch. II, §2, Exercises 17-25).

§1. Notation and Basic Definitions

An hereditary torsion theory for a commutative ring R is characterized by the family $\mathcal{F}$ of the ideals I of R for which R/I is a torsion module (for more details cf. B. Stenström's book "Rings of Quotients", Springer, Berlin 1975; Ch. VI).

It turns out that such a family $\mathcal{F}$ of ideals is the family of the neighborhoods of 0 for a certain linear topology of R.

The notion of localizing system (or topologizing system) was introduced (in a more general context) by P. Gabriel in order to characterize such topologies from an ideal-theoretic point of view (cf. Pierre Gabriel, La localisation dans les anneaux non commutatifs Exposé No. 2, in Séminaire Dubreil, Algèbre et théorie des nombres, 1959-1960; N. Bourbaki, 1961, Ch. II, §2, Exercises 17-25).

§1. Notation and Basic Definitions

An hereditary torsion theory for a commutative ring R is characterized by the family $\mathcal{F}$ of the ideals I of R for which R/I is a torsion module (for more details cf. B. Stenström's book "Rings of Quotients", Springer, Berlin 1975; Ch. VI).

It turns out that such a family $\mathcal{F}$ of ideals is the family of the neighborhoods of 0 for a certain linear topology of R.

The notion of localizing system (or topologizing system) was introduced (in a more general context) by P. Gabriel in order to characterize such topologies from an ideal-theoretic point of view (cf. Pierre Gabriel, La localisation dans les anneaux non commutatifs Exposé No. 2, in Séminaire Dubreil, Algèbre et théorie des nombres, 1959-1960; N. Bourbaki, 1961, Ch. II, §2, Exercises 17-25).

Let D be an integral domain with quotient field K. Let

- $\overline{\mathfrak{F}}(D)$ be the set of all nonzero D-submodules of K,
- $\mathfrak{F}(D)$ be the set of all nonzero fractional ideals of D, and
- $\mathbf{f}(D)$ be the set of all nonzero finitely generated D–submodules of K.

Then, obviously,

$$\mathbf{f}(D) \subseteq \mathfrak{F}(D) \subseteq \overline{\mathfrak{F}}(D).$$

Let D be an integral domain with quotient field K. Let

- $\overline{\mathfrak{F}}(D)$ be the set of all nonzero D-submodules of K,
- $\mathfrak{F}(D)$ be the set of all nonzero fractional ideals of D, and
- $\mathbf{f}(D)$ be the set of all nonzero finitely generated D–submodules of K.

Then, obviously,

$$\mathbf{f}(D) \subseteq \mathfrak{F}(D) \subseteq \overline{\mathfrak{F}}(D).$$

Let D be an integral domain with quotient field K. Let

- $\overline{\mathfrak{F}}(D)$ be the set of all nonzero D-submodules of K,
- $\mathfrak{F}(D)$ be the set of all nonzero fractional ideals of D, and
- $\mathbf{f}(D)$ be the set of all nonzero finitely generated D–submodules of K.

Then, obviously,

$$\mathbf{f}(D) \subseteq \mathfrak{F}(D) \subseteq \overline{\mathfrak{F}}(D).$$

A *localizing system $\mathcal{F}$ of an integral domain D* is a family of integral ideals of D such that

(LS1) If $I \in \mathcal{F}$ and J is an ideal of D such that $I \subseteq J$, then $J \in \mathcal{F}$;

(LS2) If $I \in \mathcal{F}$ and J is an ideal of D such that $(J :_D iD) \in \mathcal{F}$ for each $i \in I$, then $J \in \mathcal{F}$.

Note that axioms **(LS1)** and **(LS2)** ensure, in particular, that $\mathcal{F}$ is a filter: It is easy to see that if $I, J \in \mathcal{F}$, then $IJ \in \mathcal{F}$ (and, thus, $I \cap J \in \mathcal{F}$).

To avoid uninteresting cases, assume that a *localizing system $\mathcal{F}$ is nontrivial*, i.e., $(0) \notin \mathcal{F}$ and $\mathcal{F}$ is nonempty.

A *localizing system* $\mathcal{F}$ *of an integral domain* D is a family of integral ideals of D such that

(LS1) If $I \in \mathcal{F}$ and J is an ideal of D such that $I \subseteq J$, then $J \in \mathcal{F}$;

(LS2) If $I \in \mathcal{F}$ and J is an ideal of D such that $(J :_D iD) \in \mathcal{F}$ for each $i \in I$, then $J \in \mathcal{F}$.

Note that axioms **(LS1)** and **(LS2)** ensure, in particular, that $\mathcal{F}$ is a filter: It is easy to see that if $I, J \in \mathcal{F}$, then $IJ \in \mathcal{F}$ (and, thus, $I \cap J \in \mathcal{F}$).

To avoid uninteresting cases, assume that a *localizing system* $\mathcal{F}$ *is nontrivial*, i.e., $(0) \notin \mathcal{F}$ and $\mathcal{F}$ is nonempty.

A *localizing system $\mathcal{F}$ of an integral domain D* is a family of integral ideals of D such that

(LS1) If $I \in \mathcal{F}$ and J is an ideal of D such that $I \subseteq J$, then $J \in \mathcal{F}$;

(LS2) If $I \in \mathcal{F}$ and J is an ideal of D such that $(J :_D iD) \in \mathcal{F}$ for each $i \in I$, then $J \in \mathcal{F}$.

Note that axioms **(LS1)** and **(LS2)** ensure, in particular, that $\mathcal{F}$ is a filter: It is easy to see that if $I, J \in \mathcal{F}$, then $IJ \in \mathcal{F}$ (and, thus, $I \cap J \in \mathcal{F}$).

To avoid uninteresting cases, assume that a *localizing system $\mathcal{F}$ is nontrivial*, i.e., $(0) \notin \mathcal{F}$ and $\mathcal{F}$ is nonempty.

A *localizing system $\mathcal{F}$ of an integral domain D* is a family of integral ideals of D such that

(LS1) If $I \in \mathcal{F}$ and J is an ideal of D such that $I \subseteq J$, then $J \in \mathcal{F}$;

(LS2) If $I \in \mathcal{F}$ and J is an ideal of D such that $(J :_D iD) \in \mathcal{F}$ for each $i \in I$, then $J \in \mathcal{F}$.

Note that axioms **(LS1)** and **(LS2)** ensure, in particular, that $\mathcal{F}$ is a filter: It is easy to see that if $I, J \in \mathcal{F}$, then $IJ \in \mathcal{F}$ (and, thus, $I \cap J \in \mathcal{F}$).

To avoid uninteresting cases, assume that a *localizing system $\mathcal{F}$ is nontrivial*, i.e., $(0) \notin \mathcal{F}$ and $\mathcal{F}$ is nonempty.

A *localizing system $\mathcal{F}$ of an integral domain D* is a family of integral ideals of D such that

(LS1) If $I \in \mathcal{F}$ and J is an ideal of D such that $I \subseteq J$, then $J \in \mathcal{F}$;

(LS2) If $I \in \mathcal{F}$ and J is an ideal of D such that $(J :_D iD) \in \mathcal{F}$ for each $i \in I$, then $J \in \mathcal{F}$.

Note that axioms **(LS1)** and **(LS2)** ensure, in particular, that $\mathcal{F}$ is a filter: It is easy to see that if $I, J \in \mathcal{F}$, then $IJ \in \mathcal{F}$ (and, thus, $I \cap J \in \mathcal{F}$).

To avoid uninteresting cases, assume that a *localizing system $\mathcal{F}$ is nontrivial*, i.e., $(0) \notin \mathcal{F}$ and $\mathcal{F}$ is nonempty.

If $\mathcal{F}$ is a localizing system of D, then

$$D_{\mathcal{F}} := \{x \in K \mid (D :_D xD) \in \mathcal{F}\} = \bigcup\{(D : I) \mid I \in \mathcal{F}\}$$

is an overring of D called *the ring of fractions of D with respect to $\mathcal{F}$*.

and, more generally, *if E belongs to $\overline{\mathfrak{F}}(D)$,*

$$E_{\mathcal{F}} := \{x \in K \mid (E :_D xD) \in \mathcal{F}\} = \bigcup\{(E : I) \mid I \in \mathcal{F}\}$$

belongs to $\overline{\mathfrak{F}}(D_{\mathcal{F}})$.

For instance, if S is a multiplicative subset of D, then $\mathcal{F} := \{I$ ideal of $D \mid I \cap S = \emptyset\}$ is a localizing system of D and $D_{\mathcal{F}} = S^{-1}D$.

Lemma

If $\mathcal{F}$ is a localizing system of an integral domain D, then

(1) $(E \cap H)_{\mathcal{F}} = E_{\mathcal{F}} \cap H_{\mathcal{F}}$, *for each $E, H \in \overline{\mathfrak{F}}(D)$;*

(2) $(E : F)_{\mathcal{F}} = (E_{\mathcal{F}} : F_{\mathcal{F}})$, *for each $E \in \overline{\mathfrak{F}}(D)$ and for each $F \in \mathbf{f}(D)$.*

If $\mathcal{F}$ is a localizing system of D, then

$$D_{\mathcal{F}} := \{x \in K \mid (D :_D xD) \in \mathcal{F}\} = \bigcup \{(D : I) \mid I \in \mathcal{F}\}$$

is an overring of D called *the ring of fractions of D with respect to $\mathcal{F}$*.

and, more generally, *if E belongs to $\overline{\mathfrak{F}}(D)$,*

$$E_{\mathcal{F}} := \{x \in K \mid (E :_D xD) \in \mathcal{F}\} = \bigcup \{(E : I) \mid I \in \mathcal{F}\}$$

belongs to $\overline{\mathfrak{F}}(D_{\mathcal{F}})$.

For instance, if S is a multiplicative subset of D, then $\mathcal{F} := \{I \text{ ideal of } D \mid I \cap S = \emptyset\}$ is a localizing system of D and $D_{\mathcal{F}} = S^{-1}D$.

Lemma

If $\mathcal{F}$ is a localizing system of an integral domain D, then

(1) $(E \cap H)_{\mathcal{F}} = E_{\mathcal{F}} \cap H_{\mathcal{F}}$, *for each $E, H \in \overline{\mathfrak{F}}(D)$;*

(2) $(E : F)_{\mathcal{F}} = (E_{\mathcal{F}} : F_{\mathcal{F}})$, *for each $E \in \overline{\mathfrak{F}}(D)$ and for each $F \in \mathbf{f}(D)$.*

If $\mathcal{F}$ is a localizing system of D, then

$$D_{\mathcal{F}} := \{x \in K \mid (D :_D xD) \in \mathcal{F}\} = \bigcup \{(D : I) \mid I \in \mathcal{F}\}$$

is an overring of D called *the ring of fractions of D with respect to $\mathcal{F}$*.

and, more generally, *if E belongs to $\overline{\mathfrak{F}}(D)$,*

$$E_{\mathcal{F}} := \{x \in K \mid (E :_D xD) \in \mathcal{F}\} = \bigcup \{(E : I) \mid I \in \mathcal{F}\}$$

belongs to $\overline{\mathfrak{F}}(D_{\mathcal{F}})$.

For instance, if S is a multiplicative subset of D, then $\mathcal{F} := \{I$ ideal of $D \mid I \cap S = \emptyset\}$ is a localizing system of D and $D_{\mathcal{F}} = S^{-1}D$.

Lemma

If $\mathcal{F}$ is a localizing system of an integral domain D, then

(1) $(E \cap H)_{\mathcal{F}} = E_{\mathcal{F}} \cap H_{\mathcal{F}}$, *for each $E, H \in \overline{\mathfrak{F}}(D)$;*

(2) $(E : F)_{\mathcal{F}} = (E_{\mathcal{F}} : F_{\mathcal{F}})$, *for each $E \in \overline{\mathfrak{F}}(D)$ and for each $F \in \mathbf{f}(D)$.*

If $\mathcal{F}$ is a localizing system of D, then

$$D_{\mathcal{F}} := \{x \in K \mid (D :_D xD) \in \mathcal{F}\} = \bigcup \{(D : I) \mid I \in \mathcal{F}\}$$

is an overring of D called *the ring of fractions of D with respect to $\mathcal{F}$*.

and, more generally, *if E belongs to $\overline{\mathfrak{F}}(D)$,*

$$E_{\mathcal{F}} := \{x \in K \mid (E :_D xD) \in \mathcal{F}\} = \bigcup \{(E : I) \mid I \in \mathcal{F}\}$$

belongs to $\overline{\mathfrak{F}}(D_{\mathcal{F}})$.

For instance, if S is a multiplicative subset of D, then $\mathcal{F} := \{I$ ideal of $D \mid I \cap S = \emptyset\}$ is a localizing system of D and $D_{\mathcal{F}} = S^{-1}D$.

Lemma

If $\mathcal{F}$ is a localizing system of an integral domain D, then

(1) $(E \cap H)_{\mathcal{F}} = E_{\mathcal{F}} \cap H_{\mathcal{F}}$*, for each $E, H \in \overline{\mathfrak{F}}(D)$;*

(2) $(E : F)_{\mathcal{F}} = (E_{\mathcal{F}} : F_{\mathcal{F}})$*, for each $E \in \overline{\mathfrak{F}}(D)$ and for each $F \in \mathbf{f}(D)$.*

If $\mathcal{F}$ is a localizing system of D, then

$$D_{\mathcal{F}} := \{x \in K \mid (D :_D xD) \in \mathcal{F}\} = \bigcup \{(D : I) \mid I \in \mathcal{F}\}$$

is an overring of D called *the ring of fractions of D with respect to $\mathcal{F}$*.

and, more generally, *if E belongs to $\overline{\mathfrak{F}}(D)$,*

$$E_{\mathcal{F}} := \{x \in K \mid (E :_D xD) \in \mathcal{F}\} = \bigcup \{(E : I) \mid I \in \mathcal{F}\}$$

belongs to $\overline{\mathfrak{F}}(D_{\mathcal{F}})$.

For instance, if S is a multiplicative subset of D, then $\mathcal{F} := \{I$ ideal of $D \mid I \cap S = \emptyset\}$ is a localizing system of D and $D_{\mathcal{F}} = S^{-1}D$.

Lemma

If $\mathcal{F}$ is a localizing system of an integral domain D, then

(1) $(E \cap H)_{\mathcal{F}} = E_{\mathcal{F}} \cap H_{\mathcal{F}}$, *for each $E, H \in \overline{\mathfrak{F}}(D)$;*

(2) $(E : F)_{\mathcal{F}} = (E_{\mathcal{F}} : F_{\mathcal{F}})$, *for each $E \in \overline{\mathfrak{F}}(D)$ and for each $F \in \mathbf{f}(D)$.*

If $\mathcal{F}$ is a localizing system of D, then

$$D_{\mathcal{F}} := \{x \in K \mid (D :_D xD) \in \mathcal{F}\} = \bigcup\{(D : I) \mid I \in \mathcal{F}\}$$

is an overring of D called *the ring of fractions of D with respect to $\mathcal{F}$*.

and, more generally, *if E belongs to $\overline{\mathfrak{F}}(D)$,*

$$E_{\mathcal{F}} := \{x \in K \mid (E :_D xD) \in \mathcal{F}\} = \bigcup\{(E : I) \mid I \in \mathcal{F}\}$$

belongs to $\overline{\mathfrak{F}}(D_{\mathcal{F}})$.

For instance, if S is a multiplicative subset of D, then $\mathcal{F} := \{I$ ideal of $D \mid I \cap S = \emptyset\}$ is a localizing system of D and $D_{\mathcal{F}} = S^{-1}D$.

Lemma

If $\mathcal{F}$ is a localizing system of an integral domain D, then

(1) $(E \cap H)_{\mathcal{F}} = E_{\mathcal{F}} \cap H_{\mathcal{F}}$, *for each* $E, H \in \overline{\mathfrak{F}}(D)$;

(2) $(E : F)_{\mathcal{F}} = (E_{\mathcal{F}} : F_{\mathcal{F}})$, *for each* $E \in \overline{\mathfrak{F}}(D)$ *and for each* $F \in \mathbf{f}(D)$.

Localizing systems and star or semistar operations are strictly related notions.

Recall that, in 1994, Okabe and Matsuda introduced the teminology of *semistar operation* $\star$ of an integral domain D, as a natural generalization of the Krull's notion of star operation (allowing $D \neq D^\star$). However, a general notion of a "closure operation" on submodules of the total ring of fractions of a commutative ring, that includes the notion semistar operation, was previously introduced by J. Huckaba in 1988.

• A mapping $\star : \overline{\mathcal{F}}(D) \to \overline{\mathcal{F}}(D)$, $E \mapsto E^\star$ is called *a semistar operation of D* if, for all $0 \neq z \in K$ and for all $E, F \in \overline{\mathcal{F}}(D)$, the following properties hold:

$(\star_1)$ $(zE)^\star = zE^\star$;

$(\star_2)$ $E \subseteq F \Rightarrow E^\star \subseteq F^\star$;

$(\star_3)$ $E \subseteq E^\star$ and $E^{\star\star} := (E^\star)^\star = E^\star$.

Localizing systems and star or semistar operations are strictly related notions.

Recall that, in 1994, Okabe and Matsuda introduced the teminology of *semistar operation* $\star$ of an integral domain D, as a natural generalization of the Krull's notion of star operation (allowing $D \neq D^\star$).
However, a general notion of a "closure operation" on submodules of the total ring of fractions of a commutative ring, that includes the notion semistar operation, was previously introduced by J. Huckaba in 1988.

• A mapping $\star : \overline{\mathcal{F}}(D) \to \overline{\mathcal{F}}(D)$, $E \mapsto E^\star$ is called *a semistar operation of D* if, for all $0 \neq z \in K$ and for all $E, F \in \overline{\mathcal{F}}(D)$, the following properties hold:

($\star_1$) $(zE)^\star = zE^\star$;

($\star_2$) $E \subseteq F \Rightarrow E^\star \subseteq F^\star$;

($\star_3$) $E \subseteq E^\star$ and $E^{\star\star} := (E^\star)^\star = E^\star$.

Localizing systems and star or semistar operations are strictly related notions.

Recall that, in 1994, Okabe and Matsuda introduced the teminology of *semistar operation* $\star$ of an integral domain D, as a natural generalization of the Krull's notion of star operation (allowing $D \neq D^\star$). However, a general notion of a "closure operation" on submodules of the total ring of fractions of a commutative ring, that includes the notion semistar operation, was previously introduced by J. Huckaba in 1988.

• A mapping $\star : \overline{\mathcal{F}}(D) \to \overline{\mathcal{F}}(D)$, $E \mapsto E^\star$ is called *a semistar operation of D* if, for all $0 \neq z \in K$ and for all $E, F \in \overline{\mathcal{F}}(D)$, the following properties hold:

$(\star_1)$ $(zE)^\star = zE^\star$;

$(\star_2)$ $E \subseteq F \Rightarrow E^\star \subseteq F^\star$;

$(\star_3)$ $E \subseteq E^\star$ and $E^{\star\star} := (E^\star)^\star = E^\star$.

Localizing systems and star or semistar operations are strictly related notions.

Recall that, in 1994, Okabe and Matsuda introduced the teminology of *semistar operation* $\star$ of an integral domain D, as a natural generalization of the Krull's notion of star operation (allowing $D \neq D^\star$). However, a general notion of a "closure operation" on submodules of the total ring of fractions of a commutative ring, that includes the notion semistar operation, was previously introduced by J. Huckaba in 1988.

• A mapping $\star : \overline{\mathfrak{F}}(D) \to \overline{\mathfrak{F}}(D)$, $E \mapsto E^\star$ is called *a semistar operation of D* if, for all $0 \neq z \in K$ and for all $E, F \in \overline{\mathfrak{F}}(D)$, the following properties hold:

$(\star_1)$ $(zE)^\star = zE^\star$;

$(\star_2)$ $E \subseteq F \Rightarrow E^\star \subseteq F^\star$;

$(\star_3)$ $E \subseteq E^\star$ and $E^{\star\star} := (E^\star)^\star = E^\star$.

Localizing systems and star or semistar operations are strictly related notions.

Recall that, in 1994, Okabe and Matsuda introduced the teminology of *semistar operation* $\star$ of an integral domain D, as a natural generalization of the Krull's notion of star operation (allowing $D \neq D^\star$). However, a general notion of a "closure operation" on submodules of the total ring of fractions of a commutative ring, that includes the notion semistar operation, was previously introduced by J. Huckaba in 1988.

• A mapping $\star : \overline{\mathfrak{F}}(D) \to \overline{\mathfrak{F}}(D)$, $E \mapsto E^\star$ is called *a semistar operation of D* if, for all $0 \neq z \in K$ and for all $E, F \in \overline{\mathfrak{F}}(D)$, the following properties hold:

($\star_1$) $(zE)^\star = zE^\star$;

($\star_2$) $E \subseteq F \Rightarrow E^\star \subseteq F^\star$;

($\star_3$) $E \subseteq E^\star$ and $E^{\star\star} := (E^\star)^\star = E^\star$.

Localizing systems and star or semistar operations are strictly related notions.

Recall that, in 1994, Okabe and Matsuda introduced the teminology of *semistar operation* $\star$ of an integral domain D, as a natural generalization of the Krull's notion of star operation (allowing $D \neq D^\star$). However, a general notion of a "closure operation" on submodules of the total ring of fractions of a commutative ring, that includes the notion semistar operation, was previously introduced by J. Huckaba in 1988.

- A mapping $\star : \overline{\boldsymbol{\mathfrak{F}}}(D) \to \overline{\boldsymbol{\mathfrak{F}}}(D)$, $E \mapsto E^\star$ is called *a semistar operation of D* if, for all $0 \neq z \in K$ and for all $E, F \in \overline{\boldsymbol{\mathfrak{F}}}(D)$, the following properties hold:

  ($\star_1$) $(zE)^\star = zE^\star$;

  ($\star_2$) $E \subseteq F \Rightarrow E^\star \subseteq F^\star$;

  ($\star_3$) $E \subseteq E^\star$ and $E^{\star\star} := (E^\star)^\star = E^\star$.

Localizing systems and star or semistar operations are strictly related notions.

Recall that, in 1994, Okabe and Matsuda introduced the teminology of *semistar operation* $\star$ of an integral domain D, as a natural generalization of the Krull's notion of star operation (allowing $D \neq D^\star$). However, a general notion of a "closure operation" on submodules of the total ring of fractions of a commutative ring, that includes the notion semistar operation, was previously introduced by J. Huckaba in 1988.

• A mapping $\star : \overline{\mathfrak{F}}(D) \to \overline{\mathfrak{F}}(D)$, $E \mapsto E^\star$ is called *a semistar operation of* D if, for all $0 \neq z \in K$ and for all $E, F \in \overline{\mathfrak{F}}(D)$, the following properties hold:

($\star_1$) $(zE)^\star = zE^\star$;

($\star_2$) $E \subseteq F \Rightarrow E^\star \subseteq F^\star$;

($\star_3$) $E \subseteq E^\star$ and $E^{\star\star} := (E^\star)^\star = E^\star$.

• When $D^\star = D$, we say that $\star$ restricted to $\mathfrak{F}(D)$ defines *a star operation of D*

i.e., $\star : \mathfrak{F}(D) \to \mathfrak{F}(D)$ verifies the properties $(\star_2)$, $(\star_3)$ and

$$(\star\star_1) \quad (zD)^\star = zD\,, \quad (zE)^\star = zE^\star.$$

• A *semistar operation of finite type* $\star$ is an operation such that $\star = \star_f$ where

$$E^{\star_f} := \bigcup\{F^\star \mid F \subseteq E, F \in \mathbf{f}(D)\} \quad \text{for all } E \in \overline{\mathfrak{F}}(D).$$

• A *stable semistar operation* $\star$ is an operation such that

$$(E \cap H)^\star = E^\star \cap H^\star, \text{ for all } E, H \in \overline{\mathfrak{F}}(D) \quad \text{or, equivalently,}$$

$$(E : F)^\star = (E^\star : F^\star), \text{ for each } E \in \overline{\mathfrak{F}}(D) \text{ and for each } F \in \mathbf{f}(D).$$

• When $D^\star = D$, we say that $\star$ restricted to $\mathfrak{F}(D)$ defines *a star operation of D*

i.e., $\star : \mathfrak{F}(D) \to \mathfrak{F}(D)$ verifies the properties **($\star_2$)**, **($\star_3$)** and

($\star\star_1$) $(zD)^\star = zD\,, \quad (zE)^\star = zE^\star.$

• A *semistar operation of finite type* $\star$ is an operation such that $\star = \star_f$ where

$$E^{\star_f} := \bigcup\{F^\star \mid F \subseteq E, F \in \mathbf{f}(D)\} \quad \text{for all } E \in \overline{\mathfrak{F}}(D).$$

• A *stable semistar operation* $\star$ is an operation such that

$$(E \cap H)^\star = E^\star \cap H^\star, \ \text{for all } E, H \in \overline{\mathfrak{F}}(D) \quad \text{or, equivalently,}$$

$$(E : F)^\star = (E^\star : F^\star), \ \text{for each } E \in \overline{\mathfrak{F}}(D) \text{ and for each } F \in \mathbf{f}(D).$$

• When $D^\star = D$, we say that $\star$ restricted to $\mathfrak{F}(D)$ defines *a star operation of D*

i.e., $\star : \mathfrak{F}(D) \to \mathfrak{F}(D)$ verifies the properties $(\star_2)$, $(\star_3)$ and

$$(\star\star_1) \quad (zD)^\star = zD\,, \quad (zE)^\star = zE^\star.$$

• A *semistar operation of finite type* $\star$ is an operation such that $\star = \star_f$ where

$$E^{\star_f} := \bigcup\{F^\star \mid F \subseteq E, F \in \mathbf{f}(D)\} \quad \text{for all } E \in \overline{\mathfrak{F}}(D).$$

• A *stable semistar operation* $\star$ is an operation such that

$$(E \cap H)^\star = E^\star \cap H^\star, \text{ for all } E, H \in \overline{\mathfrak{F}}(D) \quad \text{or, equivalently,}$$

$$(E : F)^\star = (E^\star : F^\star), \text{ for each } E \in \overline{\mathfrak{F}}(D) \text{ and for each } F \in \mathbf{f}(D).$$

• When $D^\star = D$, we say that $\star$ restricted to $\boldsymbol{\mathcal{F}}(D)$ defines *a star operation of D*

i.e., $\star : \boldsymbol{\mathcal{F}}(D) \to \boldsymbol{\mathcal{F}}(D)$ verifies the properties **($\star_2$)**, **($\star_3$)** and

($\star\star_1$) $(zD)^\star = zD\,, \quad (zE)^\star = zE^\star$.

• A *semistar operation of finite type* $\star$ is an operation such that $\star = \star_f$ where

$$E^{\star_f} := \bigcup \{F^\star \mid F \subseteq E, F \in \mathbf{f}(D)\} \quad \text{for all } E \in \overline{\boldsymbol{\mathcal{F}}}(D).$$

• A *stable semistar operation* $\star$ is an operation such that

$$(E \cap H)^\star = E^\star \cap H^\star, \text{ for all } E, H \in \overline{\boldsymbol{\mathcal{F}}}(D) \quad \text{or, equivalently,}$$

$$(E : F)^\star = (E^\star : F^\star), \text{ for each } E \in \overline{\boldsymbol{\mathcal{F}}}(D) \text{ and for each } F \in \mathbf{f}(D).$$

§2. Localizing Systems, Module Systems and Semistar Operations

For every overring T of D the operation $\star_{\{T\}}$ *defined for all* $E \in \overline{\mathfrak{F}}(D)$ *by setting*

$$E^{\star_{\{T\}}} := ET$$

is a semistar operation of finite type.

It is straightforward that *if T is a flat D-module then $\star_{\{T\}}$ is a stable semistar operation.*

§2. Localizing Systems, Module Systems and Semistar Operations

For every overring T of D the operation $\star_{\{T\}}$ *defined for all* $E \in \overline{\boldsymbol{\mathfrak{F}}}(D)$ *by setting*

$$E^{\star_{\{T\}}} := ET$$

is a semistar operation of finite type.

It is straightforward that *if* T *is a flat* D*-module then* $\star_{\{T\}}$ *is a stable semistar operation.*

Note that, given a localizing system $\mathcal{F}$ on D, we have two canonical semistar operations in D,

- $\star_{\mathcal{F}}$ *defined, for all* $E \in \overline{\mathfrak{F}}(D)$, *by setting* $E^{\star_{\mathcal{F}}} := E_{\mathcal{F}}$;
- $\star_{\{D_{\mathcal{F}}\}}$ *defined, for all* $E \in \overline{\mathfrak{F}}(D)$, *by setting*

$$E^{\star_{\{D_{\mathcal{F}}\}}} := ED_{\mathcal{F}}\,.$$

In general, $E_{\mathcal{F}} \supseteq ED_{\mathcal{F}}$, *and maybe* $E_{\mathcal{F}} \supsetneq ED_{\mathcal{F}}$ *even if* E *is a proper integral ideal of* D. *In other words,* $\star_{\{D_{\mathcal{F}}\}} \leq \star_{\mathcal{F}}$.

For instance, let V be a valuation domain with idempotent maximal ideal M, of the type $V := K + M$, where K is a field. Let k be a proper subfield of K and define $R := k + M$. Since M is idempotent it is easy to see that $\mathcal{F} = \{M, R\}$ is a localizing system of R. Then $M_{\mathcal{F}} = R_{\mathcal{F}} = (M : M) = V$ and $MR_{\mathcal{F}} = MV = M$.

Note that, given a localizing system $\mathcal{F}$ on D , we have two canonical semistar operations in D,

- $\star_{\mathcal{F}}$ *defined, for all* $E \in \overline{\mathfrak{F}}(D)$, *by setting* $E^{\star_{\mathcal{F}}} := E_{\mathcal{F}}$;
- $\star_{\{D_{\mathcal{F}}\}}$ *defined, for all* $E \in \overline{\mathfrak{F}}(D)$, *by setting*

$$E^{\star_{\{D_{\mathcal{F}}\}}} := ED_{\mathcal{F}} .$$

In general, $E_{\mathcal{F}} \supseteq ED_{\mathcal{F}}$, *and maybe* $E_{\mathcal{F}} \supsetneq ED_{\mathcal{F}}$ *even if* E *is a proper integral ideal of* D. *In other words,* $\star_{\{D_{\mathcal{F}}\}} \leq \star_{\mathcal{F}}$.

For instance, let V be a valuation domain with idempotent maximal ideal M, of the type $V := K + M$, where K is a field. Let k be a proper subfield of K and define $R := k + M$. Since M is idempotent it is easy to see that $\mathcal{F} = \{M, R\}$ is a localizing system of R. Then $M_{\mathcal{F}} = R_{\mathcal{F}} = (M : M) = V$ and $MR_{\mathcal{F}} = MV = M$.

Note that, given a localizing system $\mathcal{F}$ on D , we have two canonical semistar operations in D,

- $\star_{\mathcal{F}}$ *defined, for all* $E \in \overline{\mathfrak{F}}(D)$*, by setting* $E^{\star_{\mathcal{F}}} := E_{\mathcal{F}}$*;*
- $\star_{\{D_{\mathcal{F}}\}}$ *defined, for all* $E \in \overline{\mathfrak{F}}(D)$*, by setting*

$$E^{\star_{\{D_{\mathcal{F}}\}}} := ED_{\mathcal{F}} \,.$$

In general, $E_{\mathcal{F}} \supseteq ED_{\mathcal{F}}$*, and maybe* $E_{\mathcal{F}} \supsetneqq ED_{\mathcal{F}}$ *even if* E *is a proper integral ideal of* D*. In other words,* $\star_{\{D_{\mathcal{F}}\}} \leq \star_{\mathcal{F}}$*.*

For instance, let V be a valuation domain with idempotent maximal ideal M, of the type $V := K + M$, where K is a field. Let k be a proper subfield of K and define $R := k + M$. Since M is idempotent it is easy to see that $\mathcal{F} = \{M, R\}$ is a localizing system of R. Then $M_{\mathcal{F}} = R_{\mathcal{F}} = (M : M) = V$ and $MR_{\mathcal{F}} = MV = M$.

Note that, given a localizing system $\mathcal{F}$ on D , we have two canonical semistar operations in D,

- $\star_{\mathcal{F}}$ *defined, for all* $E \in \overline{\mathfrak{F}}(D)$, *by setting* $E^{\star_{\mathcal{F}}} := E_{\mathcal{F}}$;
- $\star_{\{D_{\mathcal{F}}\}}$ *defined, for all* $E \in \overline{\mathfrak{F}}(D)$, *by setting*

$$E^{\star_{\{D_{\mathcal{F}}\}}} := ED_{\mathcal{F}} .$$

In general, $E_{\mathcal{F}} \supseteq ED_{\mathcal{F}}$, *and maybe* $E_{\mathcal{F}} \supsetneq ED_{\mathcal{F}}$ *even if* E *is a proper integral ideal of* D. *In other words,* $\star_{\{D_{\mathcal{F}}\}} \leq \star_{\mathcal{F}}$.

For instance, let V be a valuation domain with idempotent maximal ideal M, of the type $V := K + M$, where K is a field. Let k be a proper subfield of K and define $R := k + M$. Since M is idempotent it is easy to see that $\mathcal{F} = \{M, R\}$ is a localizing system of R. Then $M_{\mathcal{F}} = R_{\mathcal{F}} = (M : M) = V$ and $MR_{\mathcal{F}} = MV = M$.

Note that, given a localizing system $\mathcal{F}$ on D , we have two canonical semistar operations in D,

- *$\star_{\mathcal{F}}$ defined, for all $E \in \overline{\mathfrak{F}}(D)$, by setting $E^{\star_{\mathcal{F}}} := E_{\mathcal{F}}$;*
- *$\star_{\{D_{\mathcal{F}}\}}$ defined, for all $E \in \overline{\mathfrak{F}}(D)$, by setting*

$$E^{\star_{\{D_{\mathcal{F}}\}}} := ED_{\mathcal{F}} .$$

In general, *$E_{\mathcal{F}} \supseteq ED_{\mathcal{F}}$, and maybe $E_{\mathcal{F}} \supsetneqq ED_{\mathcal{F}}$ even if E is a proper integral ideal of D. In other words, $\star_{\{D_{\mathcal{F}}\}} \leq \star_{\mathcal{F}}$.*

For instance, let V be a valuation domain with idempotent maximal ideal M, of the type $V := K + M$, where K is a field. Let k be a proper subfield of K and define $R := k + M$. Since M is idempotent it is easy to see that $\mathcal{F} = \{M, R\}$ is a localizing system of R. Then $M_{\mathcal{F}} = R_{\mathcal{F}} = (M : M) = V$ and $MR_{\mathcal{F}} = MV = M$.

The following result characterizes when the equality holds.

Proposition

Let $\mathcal{F}$ be a localizing system of an integral domain D. The following are equivalent:

(i) $\star_{\{D_{\mathcal{F}}\}} = \star_{\mathcal{F}}$;

(ii) $ID_{\mathcal{F}} = I_{\mathcal{F}}$ *for each integral ideal* I *of* D;

(iii) $D_{\mathcal{F}}$ *is* D*-flat and* $\mathcal{F} = \{I \mid I$ *ideal of* D *and* $ID_{\mathcal{F}} = D_{\mathcal{F}}\}$.

The following result characterizes when the equality holds.

Proposition

Let $\mathcal{F}$ be a localizing system of an integral domain D. The following are equivalent:

(i) $\star_{\{D_{\mathcal{F}}\}} = \star_{\mathcal{F}}$;

(ii) $ID_{\mathcal{F}} = I_{\mathcal{F}}$ *for each integral ideal* I *of* D;

(iii) $D_{\mathcal{F}}$ *is* D*-flat and* $\mathcal{F} = \{I \mid I$ *ideal of* D *and* $ID_{\mathcal{F}} = D_{\mathcal{F}}\}$.

The following result characterizes when the equality holds.

Proposition

Let $\mathcal{F}$ be a localizing system of an integral domain D. The following are equivalent:

(i) $\star_{\{D_{\mathcal{F}}\}} = \star_{\mathcal{F}}$;

(ii) $ID_{\mathcal{F}} = I_{\mathcal{F}}$ *for each integral ideal* I *of* D;

(iii) $D_{\mathcal{F}}$ *is* D*-flat and* $\mathcal{F} = \{I \mid I \text{ ideal of } D \text{ and } ID_{\mathcal{F}} = D_{\mathcal{F}}\}$.

The following result characterizes when the equality holds.

Proposition

Let $\mathcal{F}$ be a localizing system of an integral domain D. The following are equivalent:

(i) $\star_{\{D_{\mathcal{F}}\}} = \star_{\mathcal{F}}$;

(ii) $ID_{\mathcal{F}} = I_{\mathcal{F}}$ *for each integral ideal I of D;*

(iii) $D_{\mathcal{F}}$ *is D-flat and* $\mathcal{F} = \{I \mid I \text{ ideal of } D \text{ and } ID_{\mathcal{F}} = D_{\mathcal{F}}\}$.

The following result characterizes when the equality holds.

Proposition

Let $\mathcal{F}$ be a localizing system of an integral domain D. The following are equivalent:

(i) $\star_{\{D_{\mathcal{F}}\}} = \star_{\mathcal{F}}$;

(ii) $ID_{\mathcal{F}} = I_{\mathcal{F}}$ *for each integral ideal* I *of* D;

(iii) $D_{\mathcal{F}}$ *is* D*-flat and* $\mathcal{F} = \{I \mid I \text{ ideal of } D \text{ and } ID_{\mathcal{F}} = D_{\mathcal{F}}\}$.

• The condition that $D_{\mathcal{F}}$ is D-flat is not equivalent to (i) and (ii) in the previous result.

Let V be a valuation domain and P a nonzero idempotent prime ideal of V, and set $\hat{\mathcal{F}}(P) := \{I \mid I \text{ ideal of } V \text{ and } I \supseteq P\}$.
Then $V_{\hat{\mathcal{F}}(P)} = V_P$ and $PV_{\hat{\mathcal{F}}(P)} = PV_P = P$. Moreover, $P_{\hat{\mathcal{F}}(P)} = (P : P) = V_P$, since $P \in \hat{\mathcal{F}}(P)$, by the previous observation. Therefore,

$$PV_{\hat{\mathcal{F}}(P)} \subsetneq P_{\hat{\mathcal{F}}(P)} = V_{\hat{\mathcal{F}}(P)} \text{ and } V_{\hat{\mathcal{F}}(P)} \text{ is obviously } V\text{-flat}$$

• The condition that $D_{\mathcal{F}}$ is D-flat is not equivalent to (i) and (ii) in the previous result.

Let V be a valuation domain and P a nonzero idempotent prime ideal of V, and set $\hat{\mathcal{F}}(P) := \{I \mid I \text{ ideal of } V \text{ and } I \supseteq P\}$.
Then $V_{\hat{\mathcal{F}}(P)} = V_P$ and $PV_{\hat{\mathcal{F}}(P)} = PV_P = P$. Moreover, $P_{\hat{\mathcal{F}}(P)} = (P : P) = V_P$, since $P \in \hat{\mathcal{F}}(P)$, by the previous observation. Therefore,

$$PV_{\hat{\mathcal{F}}(P)} \subsetneq P_{\hat{\mathcal{F}}(P)} = V_{\hat{\mathcal{F}}(P)} \text{ and } V_{\hat{\mathcal{F}}(P)} \text{ is obviously } V\text{-flat}$$

It is easy to see that *if $\star$ is a semistar operation on D, then*

$$\begin{aligned}\mathcal{F}^{\star} &:= \{I \mid I \text{ ideal of } D \text{ with } I^{\star} \cap D = D\} \\ &= \{I \mid I \text{ ideal of } D \text{ with } I^{\star} = D^{\star}\} \\ &= \{I \mid I \text{ ideal of } D \text{ with } 1 \in I^{\star}\}\end{aligned}$$

is a localizing system of D, called *the localizing system associated to $\star$*. Similarly, in case of semistar operations of finite type, we can consider the localizing system $\mathcal{F}^{\star_f}$.

On the other hand, a localizing system $\mathcal{F}$ is called *a localizing system of finite type* if for each ideal $I \in \mathcal{F}$ there exists a finitely generated ideal J of D such that $J \subseteq I$ and $J \in \mathcal{F}$. It is easy to see that, *for each localizing system $\mathcal{F}$*,

$$\mathcal{F}_f := \{I \in \mathcal{F} \mid I \supseteq J \text{ for some finitely generated ideal } J \in \mathcal{F}\}$$

is a localizing system, called *the localizing system of finite type associated to $\mathcal{F}$*.

It is easy to verify that $\mathcal{F}^{\star_f}$ *is a localizing system of finite type and* $\mathcal{F}^{\star_f} = (\mathcal{F}^{\star})_f$.

It is easy to see that *if $\star$ is a semistar operation on D, then*

$$\begin{aligned}\mathcal{F}^{\star} &:= \{I \mid I \text{ ideal of } D \text{ with } I^{\star} \cap D = D\}\\ &= \{I \mid I \text{ ideal of } D \text{ with } I^{\star} = D^{\star}\}\\ &= \{I \mid I \text{ ideal of } D \text{ with } 1 \in I^{\star}\}\end{aligned}$$

is a localizing system of D, called *the localizing system associated to $\star$*.

Similarly, in case of semistar operations of finite type, we can consider the localizing system $\mathcal{F}^{\star_f}$.

On the other hand, a localizing system $\mathcal{F}$ is called *a localizing system of finite type* if for each ideal $I \in \mathcal{F}$ there exists a finitely generated ideal J of D such that $J \subseteq I$ and $J \in \mathcal{F}$. It is easy to see that, *for each localizing system $\mathcal{F}$,*

$$\mathcal{F}_f := \{I \in \mathcal{F} \mid I \supseteq J \text{ for some finitely generated ideal } J \in \mathcal{F}\}$$

is a localizing system, called *the localizing system of finite type associated to $\mathcal{F}$*.

It is easy to verify that $\mathcal{F}^{\star_f}$ *is a localizing system of finite type and* $\mathcal{F}^{\star_f} = (\mathcal{F}^{\star})_f$.

It is easy to see that *if $\star$ is a semistar operation on D, then*

$$\begin{aligned}\mathcal{F}^{\star} &:= \{I \mid I \text{ ideal of } D \text{ with } I^{\star} \cap D = D\} \\ &= \{I \mid I \text{ ideal of } D \text{ with } I^{\star} = D^{\star}\} \\ &= \{I \mid I \text{ ideal of } D \text{ with } 1 \in I^{\star}\}\end{aligned}$$

is a localizing system of D, called *the localizing system associated to $\star$*. Similarly, in case of semistar operations of finite type, we can consider the localizing system $\mathcal{F}^{\star_f}$.

On the other hand, a localizing system $\mathcal{F}$ is called *a localizing system of finite type* if for each ideal $I \in \mathcal{F}$ there exists a finitely generated ideal J of D such that $J \subseteq I$ and $J \in \mathcal{F}$. It is easy to see that, *for each localizing system $\mathcal{F}$*,

$$\mathcal{F}_f := \{I \in \mathcal{F} \mid I \supseteq J \text{ for some finitely generated ideal } J \in \mathcal{F}\}$$

is a localizing system, called *the localizing system of finite type associated to $\mathcal{F}$*.

It is easy to verify that $\mathcal{F}^{\star_f}$ *is a localizing system of finite type and* $\mathcal{F}^{\star_f} = (\mathcal{F}^{\star})_f$.

It is easy to see that *if $\star$ is a semistar operation on D, then*

$$\begin{aligned}\mathcal{F}^{\star} &:= \{I \mid I \text{ ideal of } D \text{ with } I^{\star} \cap D = D\} \\ &= \{I \mid I \text{ ideal of } D \text{ with } I^{\star} = D^{\star}\} \\ &= \{I \mid I \text{ ideal of } D \text{ with } 1 \in I^{\star}\}\end{aligned}$$

is a localizing system of D, called *the localizing system associated to $\star$*. Similarly, in case of semistar operations of finite type, we can consider the localizing system $\mathcal{F}^{\star_f}$.

On the other hand, a localizing system $\mathcal{F}$ is called *a localizing system of finite type* if for each ideal $I \in \mathcal{F}$ there exists a finitely generated ideal J of D such that $J \subseteq I$ and $J \in \mathcal{F}$. It is easy to see that, *for each localizing system $\mathcal{F}$*,

$$\mathcal{F}_f := \{I \in \mathcal{F} \mid I \supseteq J \text{ for some finitely generated ideal } J \in \mathcal{F}\}$$

is a localizing system, called *the localizing system of finite type associated to $\mathcal{F}$*.

It is easy to verify that $\mathcal{F}^{\star_f}$ *is a localizing system of finite type and* $\mathcal{F}^{\star_f} = (\mathcal{F}^{\star})_f$.

It is easy to see that *if $\star$ is a semistar operation on D, then*

$$\begin{aligned}\mathcal{F}^\star := &\{I \mid I \text{ ideal of } D \text{ with } I^\star \cap D = D\}\\ = &\{I \mid I \text{ ideal of } D \text{ with } I^\star = D^\star\}\\ = &\{I \mid I \text{ ideal of } D \text{ with } 1 \in I^\star\}\end{aligned}$$

is a localizing system of D, called *the localizing system associated to $\star$*. Similarly, in case of semistar operations of finite type, we can consider the localizing system $\mathcal{F}^{\star_f}$.

On the other hand, a localizing system $\mathcal{F}$ is called *a localizing system of finite type* if for each ideal $I \in \mathcal{F}$ there exists a finitely generated ideal J of D such that $J \subseteq I$ and $J \in \mathcal{F}$. It is easy to see that, *for each localizing system $\mathcal{F}$*,

$$\mathcal{F}_f := \{I \in \mathcal{F} \mid I \supseteq J \text{ for some finitely generated ideal } J \in \mathcal{F}\}$$

is a localizing system, called *the localizing system of finite type associated to $\mathcal{F}$*.

It is easy to verify that $\mathcal{F}^{\star_f}$ *is a localizing system of finite type and* $\mathcal{F}^{\star_f} = (\mathcal{F}^\star)_f$.

It is easy to see that *if $\star$ is a semistar operation on D, then*

$$\begin{aligned}\mathcal{F}^{\star} := & \{I \mid I \text{ ideal of } D \text{ with } I^{\star} \cap D = D\} \\ = & \{I \mid I \text{ ideal of } D \text{ with } I^{\star} = D^{\star}\} \\ = & \{I \mid I \text{ ideal of } D \text{ with } 1 \in I^{\star}\}\end{aligned}$$

is a localizing system of D, called *the localizing system associated to $\star$*. Similarly, in case of semistar operations of finite type, we can consider the localizing system $\mathcal{F}^{\star_f}$.

On the other hand, a localizing system $\mathcal{F}$ is called *a localizing system of finite type* if for each ideal $I \in \mathcal{F}$ there exists a finitely generated ideal J of D such that $J \subseteq I$ and $J \in \mathcal{F}$. It is easy to see that, *for each localizing system $\mathcal{F}$*,

$$\mathcal{F}_f := \{I \in \mathcal{F} \mid I \supseteq J \text{ for some finitely generated ideal } J \in \mathcal{F}\}$$

is a localizing system, called *the localizing system of finite type associated to $\mathcal{F}$*.

It is easy to verify that $\mathcal{F}^{\star_f}$ *is a localizing system of finite type and* $\mathcal{F}^{\star_f} = (\mathcal{F}^{\star})_f$.

It is easy to see that *if $\star$ is a semistar operation on D, then*

$$\begin{aligned}\mathcal{F}^{\star} := & \{I \mid I \text{ ideal of } D \text{ with } I^{\star} \cap D = D\} \\ = & \{I \mid I \text{ ideal of } D \text{ with } I^{\star} = D^{\star}\} \\ = & \{I \mid I \text{ ideal of } D \text{ with } 1 \in I^{\star}\}\end{aligned}$$

is a localizing system of D, called *the localizing system associated to $\star$*. Similarly, in case of semistar operations of finite type, we can consider the localizing system $\mathcal{F}^{\star_f}$.

On the other hand, a localizing system $\mathcal{F}$ is called *a localizing system of finite type* if for each ideal $I \in \mathcal{F}$ there exists a finitely generated ideal J of D such that $J \subseteq I$ and $J \in \mathcal{F}$. It is easy to see that, *for each localizing system $\mathcal{F}$*,

$$\mathcal{F}_f := \{I \in \mathcal{F} \mid I \supseteq J \text{ for some finitely generated ideal } J \in \mathcal{F}\}$$

is a localizing system, called *the localizing system of finite type associated to $\mathcal{F}$*.

It is easy to verify that *$\mathcal{F}^{\star_f}$ is a localizing system of finite type and $\mathcal{F}^{\star_f} = (\mathcal{F}^{\star})_f$.*

Theorem

(A) *Let $\mathcal{F}$ be a localizing system of an integral domain D and let $\star_{\mathcal{F}}$ be the semistar operation on D associated with $\mathcal{F}$. Then*

$$\mathcal{F} = \mathcal{F}^{\star_{\mathcal{F}}} = \{I \text{ ideal of } D \mid I_{\mathcal{F}} \cap D = D\}.$$

(B) *Let $\star$ be a semistar operation on D and let $\mathcal{F}^{\star}$ be the localizing system associated with $\star$. Then $\star_{\mathcal{F}^{\star}} \leq \star$. Moreover,*

$$\star_{\mathcal{F}^{\star}} = \star \quad \Leftrightarrow \quad \star \text{ is a stable semistar operation.}$$

Theorem

(A) *Let $\mathcal{F}$ be a localizing system of an integral domain D and let $\star_{\mathcal{F}}$ be the semistar operation on D associated with $\mathcal{F}$. Then*

$$\mathcal{F} = \mathcal{F}^{\star_{\mathcal{F}}} = \{I \text{ ideal of } D \mid I_{\mathcal{F}} \cap D = D\}.$$

(B) *Let $\star$ be a semistar operation on D and let $\mathcal{F}^{\star}$ be the localizing system associated with $\star$. Then $\star_{\mathcal{F}^{\star}} \leq \star$. Moreover,*

$$\star_{\mathcal{F}^{\star}} = \star \quad \Leftrightarrow \quad \star \text{ is a stable semistar operation.}$$

Theorem

(A) *Let $\mathcal{F}$ be a localizing system of an integral domain D and let $\star_{\mathcal{F}}$ be the semistar operation on D associated with $\mathcal{F}$. Then*

$$\mathcal{F} = \mathcal{F}^{\star_{\mathcal{F}}} = \{I \text{ ideal of } D \mid I_{\mathcal{F}} \cap D = D\}.$$

(B) *Let $\star$ be a semistar operation on D and let $\mathcal{F}^{\star}$ be the localizing system associated with $\star$. Then $\star_{\mathcal{F}^{\star}} \leq \star$. Moreover,*

$$\star_{\mathcal{F}^{\star}} = \star \quad \Leftrightarrow \quad \star \text{ is a stable semistar operation.}$$

Both notions of *semistar operation* and *localizing system* were greatly extended in the setting of cancellative monoids.

The notion of module system introduced by Franz Halter-Koch in 2001 is a common generalization of that of ideal system (developed in Franz's book published in 1998) and that of semistar operation.

This general theory sheds new light on the connection of localizing systems with semistar operations and on a general theory of flatness and allows a new presentation of the theory of generalized integral closures. In particular, it allows a *purely multiplicative theory* of general Kronecker function rings, starting from some Lorenzen's ideas, as presented in a recent paper by F. Halter-Koch (Comm. Algebra 2015).

Both notions of *semistar operation* and *localizing system* were greatly extended in the setting of cancellative monoids.

The notion of module system introduced by Franz Halter-Koch in 2001 is a common generalization of that of ideal system (developed in Franz's book published in 1998) and that of semistar operation.

This general theory sheds new light on the connection of localizing systems with semistar operations and on a general theory of flatness and allows a new presentation of the theory of generalized integral closures. In particular, it allows a *purely multiplicative theory* of general Kronecker function rings, starting from some Lorenzen's ideas, as presented in a recent paper by F. Halter-Koch (Comm. Algebra 2015).

Both notions of *semistar operation* and *localizing system* were greatly extended in the setting of cancellative monoids.

The notion of module system introduced by Franz Halter-Koch in 2001 is a common generalization of that of ideal system (developed in Franz's book published in 1998) and that of semistar operation.

This general theory sheds new light on the connection of localizing systems with semistar operations and on a general theory of flatness and allows a new presentation of the theory of generalized integral closures. In particular, it allows a *purely multiplicative theory* of general Kronecker function rings, starting from some Lorenzen's ideas, as presented in a recent paper by F. Halter-Koch (Comm. Algebra 2015).

Both notions of *semistar operation* and *localizing system* were greatly extended in the setting of cancellative monoids.

The notion of module system introduced by Franz Halter-Koch in 2001 is a common generalization of that of ideal system (developed in Franz's book published in 1998) and that of semistar operation.

This general theory sheds new light on the connection of localizing systems with semistar operations and on a general theory of flatness and allows a new presentation of the theory of generalized integral closures. In particular, it allows a *purely multiplicative theory* of general Kronecker function rings, starting from some Lorenzen's ideas, as presented in a recent paper by F. Halter-Koch (Comm. Algebra 2015).

We need some notation.

• A *monoid* H is a multiplicative commutative semigroup with a unit element $1 \in H$ and a zero element $0 \in H$.

• $H^\bullet := H \setminus \{0\}$.

• $H^\times$ is *the group of all invertible elements of H*.

• A *groupoid* is a monoid G satisfying $G^\bullet = G^\times$.

• A *monoid H* is called *cancellative* if every $a \in H^\bullet$ is cancellative.

• Every cancellative monoid H possesses *quotient groupoid* $G \supseteq H$ (this is a groupoid G such that $G^\bullet$ is a quotient group of $H^\bullet$).

We need some notation.

• A *monoid* H is a multiplicative commutative semigroup with a unit element $1 \in H$ and a zero element $0 \in H$.

• $H^\bullet := H \setminus \{0\}$.

• $H^\times$ is *the group of all invertible elements of* H.

• A *groupoid* is a monoid G satisfying $G^\bullet = G^\times$.

• A *monoid* H is called *cancellative* if every $a \in H^\bullet$ is cancellative.

• Every cancellative monoid H possesses *quotient groupoid* $G \supseteq H$ (this is a groupoid G such that $G^\bullet$ is a quotient group of $H^\bullet$).

We need some notation.

• A *monoid* H is a multiplicative commutative semigroup with a unit element $1 \in H$ and a zero element $0 \in H$.

• $H^\bullet := H \setminus \{0\}$.

• $H^\times$ is *the group of all invertible elements of* H.

• A *groupoid* is a monoid G satisfying $G^\bullet = G^\times$.

• A *monoid* H is called *cancellative* if every $a \in H^\bullet$ is cancellative.

• Every cancellative monoid H possesses *quotient groupoid* $G \supseteq H$ (this is a groupoid G such that $G^\bullet$ is a quotient group of $H^\bullet$).

We need some notation.

• A *monoid* H is a multiplicative commutative semigroup with a unit element $1 \in H$ and a zero element $0 \in H$.

• $H^\bullet := H \setminus \{0\}$.

• $H^\times$ is *the group of all invertible elements of* H.

• A *groupoid* is a monoid G satisfying $G^\bullet = G^\times$.

• A *monoid* H is called *cancellative* if every $a \in H^\bullet$ is cancellative.

• Every cancellative monoid H possesses *quotient groupoid* $G \supseteq H$ (this is a groupoid G such that $G^\bullet$ is a quotient group of $H^\bullet$).

We need some notation.

- A *monoid* H is a multiplicative commutative semigroup with a unit element $1 \in H$ and a zero element $0 \in H$.
- $H^\bullet := H \setminus \{0\}$.
- $H^\times$ is *the group of all invertible elements of* H.
- A *groupoid* is a monoid G satisfying $G^\bullet = G^\times$.
- A *monoid* H is called *cancellative* if every $a \in H^\bullet$ is cancellative.
- Every cancellative monoid H possesses *quotient groupoid* $G \supseteq H$ (this is a groupoid G such that $G^\bullet$ is a quotient group of $H^\bullet$).

We need some notation.

• A *monoid* H is a multiplicative commutative semigroup with a unit element $1 \in H$ and a zero element $0 \in H$.

• $H^\bullet := H \setminus \{0\}$.

• $H^\times$ is *the group of all invertible elements of* H.

• A *groupoid* is a monoid G satisfying $G^\bullet = G^\times$.

• A *monoid* H is called *cancellative* if every $a \in H^\bullet$ is cancellative.

• Every cancellative monoid H possesses *quotient groupoid* $G \supseteq H$ (this is a groupoid G such that $G^\bullet$ is a quotient group of $H^\bullet$).

We need some notation.

- A *monoid* H is a multiplicative commutative semigroup with a unit element $1 \in H$ and a zero element $0 \in H$.
- $H^\bullet := H \setminus \{0\}$.
- $H^\times$ is *the group of all invertible elements of* H.
- A *groupoid* is a monoid G satisfying $G^\bullet = G^\times$.
- A *monoid* H is called *cancellative* if every $a \in H^\bullet$ is cancellative.
- Every cancellative monoid H possesses *quotient groupoid* $G \supseteq H$ (this is a groupoid G such that $G^\bullet$ is a quotient group of $H^\bullet$).

We need some notation.

- A *monoid* H is a multiplicative commutative semigroup with a unit element $1 \in H$ and a zero element $0 \in H$.
- $H^\bullet := H \setminus \{0\}$.
- $H^\times$ is *the group of all invertible elements of* H.
- A *groupoid* is a monoid G satisfying $G^\bullet = G^\times$.
- A *monoid* H is called *cancellative* if every $a \in H^\bullet$ is cancellative.
- Every cancellative monoid H possesses *quotient groupoid* $G \supseteq H$ (this is a groupoid G such that $G^\bullet$ is a quotient group of $H^\bullet$).

Let G be a groupoid and $\mathbb{P}(G)$ the power set of G.

• A *module system on* G is a map $r : \mathbb{P}(G) \to \mathbb{P}(G)$, $X \mapsto X_r$ such that the following properties are fulfilled for all $X, Y \in \mathbb{P}(G)$ and $c \in G$

(MS1) $X \cup \{0\} \subseteq X_r$;

(MS2) $X \subseteq Y_r \Rightarrow X_r \subseteq Y_r$;

(MS3) $(cX)_r = cX_r$.

• An *r-module of* G is a subset $J \subseteq G$ such that $J = J_r$ and an *r-monoid of* G is an r-module which is a submonoid of G.

Let G be a groupoid and $\mathbb{P}(G)$ the power set of G.

• A *module system on G* is a map $r : \mathbb{P}(G) \to \mathbb{P}(G)$, $X \mapsto X_r$ such that the following properties are fulfilled for all $X, Y \in \mathbb{P}(G)$ and $c \in G$

(MS1) $X \cup \{0\} \subseteq X_r$;

(MS2) $X \subseteq Y_r \Rightarrow X_r \subseteq Y_r$;

(MS3) $(cX)_r = cX_r$.

• An *r-module of G* is a subset $J \subseteq G$ such that $J = J_r$ and an *r-monoid of G* is an r-module which is a submonoid of G.

Let G be a groupoid and $\mathbb{P}(G)$ the power set of G.

• A *module system on G* is a map $r : \mathbb{P}(G) \to \mathbb{P}(G)$, $X \mapsto X_r$ such that the following properties are fulfilled for all $X, Y \in \mathbb{P}(G)$ and $c \in G$

(MS1) $X \cup \{0\} \subseteq X_r$;

(MS2) $X \subseteq Y_r \Rightarrow X_r \subseteq Y_r$;

(MS3) $(cX)_r = cX_r$.

• An *r-module of G* is a subset $J \subseteq G$ such that $J = J_r$ and an *r-monoid of G* is an r-module which is a submonoid of G.

Let G be a groupoid and $\mathbb{P}(G)$ the power set of G.

• A *module system on G* is a map $r : \mathbb{P}(G) \to \mathbb{P}(G)$, $X \mapsto X_r$ such that the following properties are fulfilled for all $X, Y \in \mathbb{P}(G)$ and $c \in G$

(MS1) $X \cup \{0\} \subseteq X_r$;

(MS2) $X \subseteq Y_r \Rightarrow X_r \subseteq Y_r$;

(MS3) $(cX)_r = cX_r$.

• An *r-module of G* is a subset $J \subseteq G$ such that $J = J_r$ and an *r-monoid of G* is an r-module which is a submonoid of G.

Let G be a groupoid and $\mathbb{P}(G)$ the power set of G.

• A *module system on G* is a map $r : \mathbb{P}(G) \to \mathbb{P}(G)$, $X \mapsto X_r$ such that the following properties are fulfilled for all $X, Y \in \mathbb{P}(G)$ and $c \in G$

(MS1) $X \cup \{0\} \subseteq X_r$;

(MS2) $X \subseteq Y_r \Rightarrow X_r \subseteq Y_r$;

(MS3) $(cX)_r = cX_r$.

• An *r-module of G* is a subset $J \subseteq G$ such that $J = J_r$ and an *r-monoid of G* is an r-module which is a submonoid of G.

Let G be a groupoid and $\mathbb{P}(G)$ the power set of G.

• A *module system on G* is a map $r : \mathbb{P}(G) \to \mathbb{P}(G)$, $X \mapsto X_r$ such that the following properties are fulfilled for all $X, Y \in \mathbb{P}(G)$ and $c \in G$

(MS1) $X \cup \{0\} \subseteq X_r$;

(MS2) $X \subseteq Y_r \;\Rightarrow\; X_r \subseteq Y_r$;

(MS3) $(cX)_r = cX_r$.

• An *r-module of G* is a subset $J \subseteq G$ such that $J = J_r$ and an *r-monoid of G* is an r-module which is a submonoid of G.

Let r be a module system on G and H a submonoid of G.

- The map $r[H] : \mathbb{P}(G) \to \mathbb{P}(G)$, $X \mapsto (XH)_r$ is called *the module system (on G) extension of r with H* and it is easy to see that *$r = r[H]$ if and only if $H \subseteq \{1\}_r$.*

- If H is an r-monoid, submonoid of G, then $r_H := r[H]|_{\mathbb{P}(H)} : \mathbb{P}(H) \to \mathbb{P}(H)$, $X \mapsto (XH)_r$ is an "usual" ideal system on H called *the ideal system induced by r on H.*

- Disregarding the additive structure, a field (respectively, an integral domain) is a groupoid (respectively, a cancellative monoid). In this particular situation, *the notion of module system (respectively, ideal system) corresponds –in a natural way– to the notion of semistar* (respectively, *star*) *operation.*

Let r be a module system on G and H a submonoid of G.

• The map $r[H] : \mathbb{P}(G) \to \mathbb{P}(G)$, $X \mapsto (XH)_r$ is called *the module system (on G) extension of r with H* and it is easy to see that *$r = r[H]$ if and only if $H \subseteq \{1\}_r$.*

• If H is an r-monoid, submonoid of G, then $r_H := r[H]|_{\mathbb{P}(H)} : \mathbb{P}(H) \to \mathbb{P}(H)$, $X \mapsto (XH)_r$ is an "usual" ideal system on H called *the ideal system induced by r on H.*

• Disregarding the additive structure, a field (respectively, an integral domain) is a groupoid (respectively, a cancellative monoid). In this particular situation, *the notion of module system (respectively, ideal system) corresponds –in a natural way– to the notion of semistar* (respectively, *star*) *operation.*

Let r be a module system on G and H a submonoid of G.

- The map $r[H] : \mathbb{P}(G) \to \mathbb{P}(G)$, $X \mapsto (XH)_r$ is called *the module system (on G) extension of r with H* and it is easy to see that *$r = r[H]$ if and only if $H \subseteq \{1\}_r$.*

- If H is an r-monoid, submonoid of G, then $r_H := r[H]|_{\mathbb{P}(H)} : \mathbb{P}(H) \to \mathbb{P}(H)$, $X \mapsto (XH)_r$ is an "usual" ideal system on H called *the ideal system induced by r on H.*

- Disregarding the additive structure, a field (respectively, an integral domain) is a groupoid (respectively, a cancellative monoid). In this particular situation, *the notion of module system (respectively, ideal system) corresponds –in a natural way– to the notion of semistar (respectively, star) operation.*

Let r be a module system on G and H a submonoid of G.

• The map $r[H] : \mathbb{P}(G) \to \mathbb{P}(G)$, $X \mapsto (XH)_r$ is called *the module system (on G) extension of r with H* and it is easy to see that *$r = r[H]$ if and only if $H \subseteq \{1\}_r$.*

• If H is an r-monoid, submonoid of G, then $r_H := r[H]|_{\mathbb{P}(H)} : \mathbb{P}(H) \to \mathbb{P}(H)$, $X \mapsto (XH)_r$ is an "usual" ideal system on H called *the ideal system induced by r on H.*

• Disregarding the additive structure, a field (respectively, an integral domain) is a groupoid (respectively, a cancellative monoid). In this particular situation, *the notion of module system (respectively, ideal system) corresponds –in a natural way– to the notion of semistar* (respectively, *star*) *operation.*

Let r be a module system on G and H a submonoid of G.

• The map $r[H] : \mathbb{P}(G) \to \mathbb{P}(G)$, $X \mapsto (XH)_r$ is called *the module system (on G) extension of r with H* and it is easy to see that *$r = r[H]$ if and only if $H \subseteq \{1\}_r$.*

• If H is an r-monoid, submonoid of G, then $r_H := r[H]|_{\mathbb{P}(H)} : \mathbb{P}(H) \to \mathbb{P}(H)$, $X \mapsto (XH)_r$ is an "usual" ideal system on H called *the ideal system induced by r on H.*

• Disregarding the additive structure, a field (respectively, an integral domain) is a groupoid (respectively, a cancellative monoid). In this particular situation, *the notion of module system (respectively, ideal system) corresponds –in a natural way– to the notion of semistar (respectively, star) operation*.

Let r be a module system on G, H a submonoid of G and $\mathbb{P}_f(X)$ the set of all finite subsets of a subset X of G.

• The map $r_f : \mathbb{P}(G) \to \mathbb{P}(G)$, $X \mapsto X_{r_f} := \bigcup\{E_r \mid E \in \mathbb{P}_f(X)\}$ is a module system called *the module system of finite type associated to r* ; r is called *a module system of finite type if $r = r_f$*.

• If H is an r-monoid, submonoid of G and if r is a module system of *finite type* then $r_H := r[H]|_{\mathbb{P}(H)} : \mathbb{P}(H) \to \mathbb{P}(H)$, $X \mapsto (XH)_r$ is an ideal system of *finite type* on H.

Let r be a module system on G, H a submonoid of G and $\mathbb{P}_f(X)$ the set of all finite subsets of a subset X of G.

- The map $r_f : \mathbb{P}(G) \to \mathbb{P}(G)$, $X \mapsto X_{r_f} := \bigcup\{E_r \mid E \in \mathbb{P}_f(X)\}$ is a module system called *the module system of finite type associated to* r ; r is called *a module system of finite type if* $r = r_f$.

- If H is an r-monoid, submonoid of G and if r is a module system of *finite type* then $r_H := r[H]|_{\mathbb{P}(H)} : \mathbb{P}(H) \to \mathbb{P}(H)$, $X \mapsto (XH)_r$ is an ideal system of *finite type* on H.

Let r be a module system on G, H a submonoid of G and $\mathbb{P}_f(X)$ the set of all finite subsets of a subset X of G.

• The map $r_f : \mathbb{P}(G) \to \mathbb{P}(G)$, $X \mapsto X_{r_f} := \bigcup\{E_r \mid E \in \mathbb{P}_f(X)\}$ is a module system called *the module system of finite type associated to r* ;
r is called *a module system of finite type if $r = r_f$*.

• If H is an r-monoid, submonoid of G and if r is a module system of *finite type* then $r_H := r[H]|_{\mathbb{P}(H)} : \mathbb{P}(H) \to \mathbb{P}(H)$, $X \mapsto (XH)_r$ is an ideal system of *finite type* on H.

Let r be a module system on G, H a submonoid of G and $\mathbb{P}_f(X)$ the set of all finite subsets of a subset X of G.

• The map $r_f : \mathbb{P}(G) \to \mathbb{P}(G)$, $X \mapsto X_{r_f} := \bigcup\{E_r \mid E \in \mathbb{P}_f(X)\}$ is a module system called *the module system of finite type associated to r* ; r is called *a module system of finite type if $r = r_f$*.

• If H is an r-monoid, submonoid of G and if r is a module system of *finite type* then $r_H := r[H]|_{\mathbb{P}(H)} : \mathbb{P}(H) \to \mathbb{P}(H)$, $X \mapsto (XH)_r$ is an ideal system of *finite type* on H.

Let r be a module system on G, H a submonoid of G and $\mathbb{P}_f(X)$ the set of all finite subsets of a subset X of G.

• The map $r_f : \mathbb{P}(G) \to \mathbb{P}(G)$, $X \mapsto X_{r_f} := \bigcup\{E_r \mid E \in \mathbb{P}_f(X)\}$ is a module system called *the module system of finite type associated to r* ; r is called *a module system of finite type if $r = r_f$*.

• If H is an r-monoid, submonoid of G and if r is a module system of *finite type* then $r_H := r[H]|_{\mathbb{P}(H)} : \mathbb{P}(H) \to \mathbb{P}(H)$, $X \mapsto (XH)_r$ is an ideal system of *finite type* on H.

• Let H be a cancellative monoid and let q be an ideal system of finite type on H. Denote by $\mathfrak{I}_q$ the set of q-ideals of H and define a *q-multiplication of q-ideals* by setting $I \cdot_q J := (IJ)_q$.

A subset $\mathcal{L} \subseteq \mathfrak{I}_q$ is called a *q-localizing system on H* if

(q-LS1) If $I \in \mathcal{L}$ and $J \in \mathfrak{I}_q$ is such that $I \subseteq J$, then $J \in \mathcal{L}$;

(q-LS2) If $I \in \mathcal{L}$ and $J \in \mathfrak{I}_q$ such that $(J : iH) \in \mathcal{L}$ for each $i \in I$, then $J \in \mathcal{L}$.

Proposition

If $\mathcal{L}$ is q-localizing system on a cancellative monoid H having G as a quotient groupoid, then

• *the map $\rho_{\mathcal{L}} : \mathbb{P}(G) \to \mathbb{P}(G)$, $X \mapsto X_{\mathcal{L}} := \bigcup\{(X_q : L) \mid L \in \mathcal{L}\} = \{y \in G \mid (X_q :_H y) \in \mathcal{L}\}$ is a module system on G, called the* module system induced by $\mathcal{L}$.

• *the map $\rho_{\mathcal{L}}|_{\mathbb{P}(H_{\mathcal{L}})} : \mathbb{P}(H_{\mathcal{L}}) \to \mathbb{P}(H_{\mathcal{L}})$ is an ideal system on $H_{\mathcal{L}}$.*

• Let H be a cancellative monoid and let q be an ideal system of finite type on H. Denote by $\mathfrak{I}_q$ the set of q-ideals of H and define a *q-multiplication of q-ideals* by setting $I \cdot_q J := (IJ)_q$.

A subset $\mathcal{L} \subseteq \mathfrak{I}_q$ is called a *q-localizing system on H* if
(q-LS1) If $I \in \mathcal{L}$ and $J \in \mathfrak{I}_q$ is such that $I \subseteq J$, then $J \in \mathcal{L}$;
(q-LS2) If $I \in \mathcal{L}$ and $J \in \mathfrak{I}_q$ such that $(J : iH) \in \mathcal{L}$ for each $i \in I$, then $J \in \mathcal{L}$.

Proposition

If $\mathcal{L}$ is q-localizing system on a cancellative monoid H having G as a quotient groupoid, then

• *the map $\rho_{\mathcal{L}} : \mathbb{P}(G) \to \mathbb{P}(G)$, $X \mapsto X_{\mathcal{L}} := \bigcup\{(X_q : L) \mid L \in \mathcal{L}\} = \{y \in G \mid (X_q :_H y) \in \mathcal{L}\}$ is a module system on G, called the* module system induced by $\mathcal{L}$.

• *the map $\rho_{\mathcal{L}}|_{\mathbb{P}(H_{\mathcal{L}})} : \mathbb{P}(H_{\mathcal{L}}) \to \mathbb{P}(H_{\mathcal{L}})$ is an ideal system on $H_{\mathcal{L}}$.*

• Let H be a cancellative monoid and let q be an ideal system of finite type on H. Denote by $\mathfrak{I}_q$ the set of q-ideals of H and define a *q-multiplication of q-ideals* by setting $I \cdot_q J := (IJ)_q$.

A subset $\mathcal{L} \subseteq \mathfrak{I}_q$ is called a *q-localizing system on H* if
(q-LS1) If $I \in \mathcal{L}$ and $J \in \mathfrak{I}_q$ is such that $I \subseteq J$, then $J \in \mathcal{L}$;
(q-LS2) If $I \in \mathcal{L}$ and $J \in \mathfrak{I}_q$ such that $(J : iH) \in \mathcal{L}$ for each $i \in I$, then $J \in \mathcal{L}$.

Proposition

If $\mathcal{L}$ is q-localizing system on a cancellative monoid H having G as a quotient groupoid, then

• *the map $\rho_{\mathcal{L}} : \mathbb{P}(G) \to \mathbb{P}(G)$, $X \mapsto X_{\mathcal{L}} := \bigcup\{(X_q : L) \mid L \in \mathcal{L}\} = \{y \in G \mid (X_q :_H y) \in \mathcal{L}\}$ is a module system on G, called the* module system induced by $\mathcal{L}$.

• *the map $\rho_{\mathcal{L}}|_{\mathbb{P}(H_{\mathcal{L}})} : \mathbb{P}(H_{\mathcal{L}}) \to \mathbb{P}(H_{\mathcal{L}})$ is an ideal system on $H_{\mathcal{L}}$.*

• Let H be a cancellative monoid and let q be an ideal system of finite type on H. Denote by $\mathfrak{I}_q$ the set of q-ideals of H and define a *q-multiplication of q-ideals* by setting $I \cdot_q J := (IJ)_q$.

A subset $\mathcal{L} \subseteq \mathfrak{I}_q$ is called a *q-localizing system on H* if
(q-LS1) If $I \in \mathcal{L}$ and $J \in \mathfrak{I}_q$ is such that $I \subseteq J$, then $J \in \mathcal{L}$;
(q-LS2) If $I \in \mathcal{L}$ and $J \in \mathfrak{I}_q$ such that $(J : iH) \in \mathcal{L}$ for each $i \in I$, then $J \in \mathcal{L}$.

Proposition

If $\mathcal{L}$ is q-localizing system on a cancellative monoid H having G as a quotient groupoid, then

• *the map $\rho_{\mathcal{L}} : \mathbb{P}(G) \to \mathbb{P}(G)$, $X \mapsto X_{\mathcal{L}} := \bigcup\{(X_q : L) \mid L \in \mathcal{L}\} = \{y \in G \mid (X_q :_H y) \in \mathcal{L}\}$ is a module system on G, called the* module system induced by $\mathcal{L}$.

• *the map $\rho_{\mathcal{L}}|_{\mathbb{P}(H_{\mathcal{L}})} : \mathbb{P}(H_{\mathcal{L}}) \to \mathbb{P}(H_{\mathcal{L}})$ is an ideal system on $H_{\mathcal{L}}$.*

• Let H be a cancellative monoid and let q be an ideal system of finite type on H. Denote by $\mathfrak{I}_q$ the set of q-ideals of H and define a *q-multiplication of q-ideals* by setting $I \cdot_q J := (IJ)_q$.

A subset $\mathcal{L} \subseteq \mathfrak{I}_q$ is called a *q-localizing system on H* if

(q-LS1) If $I \in \mathcal{L}$ and $J \in \mathfrak{I}_q$ is such that $I \subseteq J$, then $J \in \mathcal{L}$;

(q-LS2) If $I \in \mathcal{L}$ and $J \in \mathfrak{I}_q$ such that $(J : iH) \in \mathcal{L}$ for each $i \in I$, then $J \in \mathcal{L}$.

Proposition

If $\mathcal{L}$ is q-localizing system on a cancellative monoid H having G as a quotient groupoid, then

• *the map $\rho_{\mathcal{L}} : \mathbb{P}(G) \to \mathbb{P}(G)$, $X \mapsto X_{\mathcal{L}} := \bigcup\{(X_q : L) \mid L \in \mathcal{L}\} = \{y \in G \mid (X_q :_H y) \in \mathcal{L}\}$ is a module system on G, called the* module system induced by $\mathcal{L}$.

• *the map $\rho_{\mathcal{L}}|_{\mathbb{P}(H_{\mathcal{L}})} : \mathbb{P}(H_{\mathcal{L}}) \to \mathbb{P}(H_{\mathcal{L}})$ is an ideal system on $H_{\mathcal{L}}$.*

• Let H be a cancellative monoid and let q be an ideal system of finite type on H. Denote by $\mathfrak{I}_q$ the set of q-ideals of H and define a *q-multiplication of q-ideals* by setting $I \cdot_q J := (IJ)_q$.

A subset $\mathcal{L} \subseteq \mathfrak{I}_q$ is called a *q-localizing system on H* if

(q-LS1) If $I \in \mathcal{L}$ and $J \in \mathfrak{I}_q$ is such that $I \subseteq J$, then $J \in \mathcal{L}$;

(q-LS2) If $I \in \mathcal{L}$ and $J \in \mathfrak{I}_q$ such that $(J : iH) \in \mathcal{L}$ for each $i \in I$, then $J \in \mathcal{L}$.

Proposition

If $\mathcal{L}$ is q-localizing system on a cancellative monoid H having G as a quotient groupoid, then

• *the map $\rho_{\mathcal{L}} : \mathbb{P}(G) \to \mathbb{P}(G)$, $X \mapsto X_{\mathcal{L}} := \bigcup\{(X_q : L) \mid L \in \mathcal{L}\} = \{y \in G \mid (X_q :_H y) \in \mathcal{L}\}$ is a module system on G, called the* module system induced by $\mathcal{L}$.

• *the map $\rho_{\mathcal{L}}|_{\mathbb{P}(H_{\mathcal{L}})} : \mathbb{P}(H_{\mathcal{L}}) \to \mathbb{P}(H_{\mathcal{L}})$ is an ideal system on $H_{\mathcal{L}}$.*

• Let H be a cancellative monoid and let q be an ideal system of finite type on H. Denote by $\mathfrak{I}_q$ the set of q-ideals of H and define a *q-multiplication of q-ideals* by setting $I \cdot_q J := (IJ)_q$.

A subset $\mathcal{L} \subseteq \mathfrak{I}_q$ is called a *q-localizing system on H* if
(q-LS1) If $I \in \mathcal{L}$ and $J \in \mathfrak{I}_q$ is such that $I \subseteq J$, then $J \in \mathcal{L}$;
(q-LS2) If $I \in \mathcal{L}$ and $J \in \mathfrak{I}_q$ such that $(J : iH) \in \mathcal{L}$ for each $i \in I$, then $J \in \mathcal{L}$.

Proposition

If $\mathcal{L}$ is q-localizing system on a cancellative monoid H having G as a quotient groupoid, then

• *the map $\rho_{\mathcal{L}} : \mathbb{P}(G) \to \mathbb{P}(G)$, $X \mapsto X_{\mathcal{L}} := \bigcup\{(X_q : L) \mid L \in \mathcal{L}\} = \{y \in G \mid (X_q :_H y) \in \mathcal{L}\}$ is a module system on G, called the module system induced by $\mathcal{L}$.*

• *the map $\rho_{\mathcal{L}}|_{\mathbb{P}(H_{\mathcal{L}})} : \mathbb{P}(H_{\mathcal{L}}) \to \mathbb{P}(H_{\mathcal{L}})$ is an ideal system on $H_{\mathcal{L}}$.*

• Let H be a cancellative monoid and let q be an ideal system of finite type on H. Denote by $\mathfrak{I}_q$ the set of q-ideals of H and define a *q-multiplication of q-ideals* by setting $I \cdot_q J := (IJ)_q$.

A subset $\mathcal{L} \subseteq \mathfrak{I}_q$ is called a *q-localizing system on H* if
(q-LS1) If $I \in \mathcal{L}$ and $J \in \mathfrak{I}_q$ is such that $I \subseteq J$, then $J \in \mathcal{L}$;
(q-LS2) If $I \in \mathcal{L}$ and $J \in \mathfrak{I}_q$ such that $(J : iH) \in \mathcal{L}$ for each $i \in I$, then $J \in \mathcal{L}$.

Proposition

If $\mathcal{L}$ is q-localizing system on a cancellative monoid H having G as a quotient groupoid, then

• *the map* $\rho_{\mathcal{L}} : \mathbb{P}(G) \to \mathbb{P}(G)$, $X \mapsto X_{\mathcal{L}} := \bigcup\{(X_q : L) \mid L \in \mathcal{L}\} = \{y \in G \mid (X_q :_H y) \in \mathcal{L}\}$ *is a module system on* G, *called the* module system induced by $\mathcal{L}$.

• *the map* $\rho_{\mathcal{L}}|_{\mathbb{P}(H_{\mathcal{L}})} : \mathbb{P}(H_{\mathcal{L}}) \to \mathbb{P}(H_{\mathcal{L}})$ *is an ideal system on* $H_{\mathcal{L}}$.

• Let H be a cancellative monoid and let q be an ideal system of finite type on H. Denote by $\mathfrak{I}_q$ the set of q-ideals of H and define a *q-multiplication of q-ideals* by setting $I \cdot_q J := (IJ)_q$.

A subset $\mathcal{L} \subseteq \mathfrak{I}_q$ is called a *q-localizing system on H* if
(q-LS1) If $I \in \mathcal{L}$ and $J \in \mathfrak{I}_q$ is such that $I \subseteq J$, then $J \in \mathcal{L}$;
(q-LS2) If $I \in \mathcal{L}$ and $J \in \mathfrak{I}_q$ such that $(J : iH) \in \mathcal{L}$ for each $i \in I$, then $J \in \mathcal{L}$.

Proposition

If $\mathcal{L}$ is q-localizing system on a cancellative monoid H having G as a quotient groupoid, then

• *the map* $\rho_{\mathcal{L}} : \mathbb{P}(G) \to \mathbb{P}(G)$, $X \mapsto X_{\mathcal{L}} := \bigcup\{(X_q : L) \mid L \in \mathcal{L}\} = \{y \in G \mid (X_q :_H y) \in \mathcal{L}\}$ *is a module system on G, called the* module system induced by $\mathcal{L}$.

• *the map* $\rho_{\mathcal{L}}|_{\mathbb{P}(H_{\mathcal{L}})} : \mathbb{P}(H_{\mathcal{L}}) \to \mathbb{P}(H_{\mathcal{L}})$ *is an ideal system on* $H_{\mathcal{L}}$.

- Let H be a cancellative monoid, G its quotient groupoid, let q denote an ideal system of finite type on H and r a module system on G such that $q \leq r$. The *module system r is called q-stable if $(I \cap J)_r = I_r \cap J_r$ for all q-modules I and J.*

Next result provides an example of the general statements obtained by Franz Halter-Koch in the module systems setting:

Theorem, Halter-Koch, 2001

Let H be a cancellative monoid, G its quotient groupoid, let q denote an ideal system of finite type on H and r a module system on G, $q \leq r$.

- *If $\mathbf{LS}_q(H)$ denotes the set of all q-localizing systems on H and $\mathbf{ModSys}(G)$ the set of all module systems on G, then the canonical map $\rho : \mathbf{LS}_q(H) \to \mathbf{ModSys}(G)$, $\mathcal{L} \mapsto \rho_{\mathcal{L}}$ is injective and order preserving.*
- *The image of this map is the set $\{r$ is a module system on $G \mid r$ is q-stable and $q \leq r = \rho_\Lambda\}$, where $\Lambda := \Lambda_{q,r} := \{I \in \mathfrak{I}_q(H) \mid 1 \in I_r\}$ is the q-localizing system associated to r (and q).*

- Let H be a cancellative monoid, G its quotient groupoid, let q denote an ideal system of finite type on H and r a module system on G such that $q \leq r$. The *module system r is called q-stable if $(I \cap J)_r = I_r \cap J_r$ for all q-modules I and J.*

Next result provides an example of the general statements obtained by Franz Halter-Koch in the module systems setting:

Theorem, Halter-Koch, 2001

Let H be a cancellative monoid, G its quotient groupoid, let q denote an ideal system of finite type on H and r a module system on G, $q \leq r$.

- *If $\mathbf{LS}_q(H)$ denotes the set of all q-localizing systems on H and $\mathbf{ModSys}(G)$ the set of all module systems on G, then the canonical map $\rho : \mathbf{LS}_q(H) \to \mathbf{ModSys}(G)$, $\mathcal{L} \mapsto \rho_{\mathcal{L}}$ is injective and order preserving.*
- *The image of this map is the set $\{r$ is a module system on $G \mid r$ is q-stable and $q \leq r = \rho_\Lambda\}$, where $\Lambda := \Lambda_{q,r} := \{I \in \mathfrak{I}_q(H) \mid 1 \in I_r\}$ is the q-localizing system associated to r (and q).*

- Let H be a cancellative monoid, G its quotient groupoid, let q denote an ideal system of finite type on H and r a module system on G such that $q \leq r$. The *module system r is called q-stable if $(I \cap J)_r = I_r \cap J_r$ for all q-modules I and J.*

Next result provides an example of the general statements obtained by Franz Halter-Koch in the module systems setting:

Theorem, Halter-Koch, 2001

Let H be a cancellative monoid, G its quotient groupoid, let q denote an ideal system of finite type on H and r a module system on G, $q \leq r$.

- *If $\mathbf{LS}_q(H)$ denotes the set of all q-localizing systems on H and $\mathbf{ModSys}(G)$ the set of all module systems on G, then the canonical map $\rho : \mathbf{LS}_q(H) \to \mathbf{ModSys}(G)$, $\mathcal{L} \mapsto \rho_{\mathcal{L}}$ is injective and order preserving.*
- *The image of this map is the set $\{r$ is a module system on $G \mid r$ is q-stable and $q \leq r = \rho_\Lambda\}$, where $\Lambda := \Lambda_{q,r} := \{I \in \mathfrak{J}_q(H) \mid 1 \in I_r\}$ is the q-localizing system associated to r (and q).*

- Let H be a cancellative monoid, G its quotient groupoid, let q denote an ideal system of finite type on H and r a module system on G such that $q \leq r$. The *module system r is called q-stable if $(I \cap J)_r = I_r \cap J_r$ for all q-modules I and J.*

Next result provides an example of the general statements obtained by Franz Halter-Koch in the module systems setting:

Theorem, Halter-Koch, 2001

Let H be a cancellative monoid, G its quotient groupoid, let q denote an ideal system of finite type on H and r a module system on G, $q \leq r$.

- *If $\mathbf{LS}_q(H)$ denotes the set of all q-localizing systems on H and $\mathbf{ModSys}(G)$ the set of all module systems on G, then the canonical map $\rho : \mathbf{LS}_q(H) \to \mathbf{ModSys}(G)$, $\mathcal{L} \mapsto \rho_{\mathcal{L}}$ is injective and order preserving.*
- *The image of this map is the set $\{r$ is a module system on $G \mid r$ is q-stable and $q \leq r = \rho_\Lambda\}$, where $\Lambda := \Lambda_{q,r} := \{I \in \mathfrak{I}_q(H) \mid 1 \in I_r\}$ is the q-localizing system associated to r (and q).*

• Let H be a cancellative monoid, G its quotient groupoid, let q denote an ideal system of finite type on H and r a module system on G such that $q \leq r$. The *module system r is called q-stable if $(I \cap J)_r = I_r \cap J_r$ for all q-modules I and J.*

Next result provides an example of the general statements obtained by Franz Halter-Koch in the module systems setting:

Theorem, Halter-Koch, 2001

Let H be a cancellative monoid, G its quotient groupoid, let q denote an ideal system of finite type on H and r a module system on G, $q \leq r$.

• *If $\mathbf{LS}_q(H)$ denotes the set of all q-localizing systems on H and $\mathbf{ModSys}(G)$ the set of all module systems on G, then the canonical map $\rho : \mathbf{LS}_q(H) \to \mathbf{ModSys}(G)$, $\mathcal{L} \mapsto \rho_{\mathcal{L}}$ is injective and order preserving.*

• *The image of this map is the set $\{r$ is a module system on $G \mid r$ is q-stable and $q \leq r = \rho_\Lambda\}$, where $\Lambda := \Lambda_{q,r} := \{I \in \mathfrak{I}_q(H) \mid 1 \in I_r\}$ is the q-localizing system associated to r (and q).*

• Let H be a cancellative monoid, G its quotient groupoid, let q denote an ideal system of finite type on H and r a module system on G such that $q \leq r$. The *module system r is called q-stable if $(I \cap J)_r = I_r \cap J_r$ for all q-modules I and J.*

Next result provides an example of the general statements obtained by Franz Halter-Koch in the module systems setting:

Theorem, Halter-Koch, 2001

Let H be a cancellative monoid, G its quotient groupoid, let q denote an ideal system of finite type on H and r a module system on G, $q \leq r$.

• *If $\mathbf{LS}_q(H)$ denotes the set of all q-localizing systems on H and $\mathbf{ModSys}(G)$ the set of all module systems on G, then the canonical map $\rho : \mathbf{LS}_q(H) \to \mathbf{ModSys}(G)$, $\mathcal{L} \mapsto \rho_{\mathcal{L}}$ is injective and order preserving.*

• *The image of this map is the set $\{r$ is a module system on $G \mid r$ is q-stable and $q \leq r = \rho_\Lambda\}$, where $\Lambda := \Lambda_{q,r} := \{I \in \mathfrak{I}_q(H) \mid 1 \in I_r\}$ is the q-localizing system associated to r (and q).*

- Let H be a cancellative monoid, G its quotient groupoid, let q denote an ideal system of finite type on H and r a module system on G such that $q \leq r$. The *module system r is called q-stable if $(I \cap J)_r = I_r \cap J_r$ for all q-modules I and J.*

Next result provides an example of the general statements obtained by Franz Halter-Koch in the module systems setting:

Theorem, Halter-Koch, 2001

Let H be a cancellative monoid, G its quotient groupoid, let q denote an ideal system of finite type on H and r a module system on G, $q \leq r$.

- *If $\mathbf{LS}_q(H)$ denotes the set of all q-localizing systems on H and $\mathbf{ModSys}(G)$ the set of all module systems on G, then the canonical map $\rho : \mathbf{LS}_q(H) \to \mathbf{ModSys}(G)$, $\mathcal{L} \mapsto \rho_{\mathcal{L}}$ is injective and order preserving.*
- *The image of this map is the set $\{r$ is a module system on G $\mid$ r is q-stable and $q \leq r = \rho_\Lambda\}$, where $\Lambda := \Lambda_{q,r} := \{I \in \mathfrak{I}_q(H) \mid 1 \in I_r\}$ is the q-localizing system associated to r (and q).*

§3. Halter-Koch's axiomatic approach to a general version of the Kronecker function ring and spaces of valuation domains

Toward the middle of the XIXth century, E.E. Kummer discovered that the ring of integers of a cyclotomic field does not have the unique factorization property.

Few years later, in 1847 Kummer introduced the concept of "ideal numbers" to re-establish some of the factorization theory for cyclotomic integers with prime exponents. (In 1856 he generalized his theory to the case of cyclotomic integers with arbitrary exponents.)

R. Dedekind in 1871 (XI supplement to Dirichlet's "Vorlesungen über Zahlentheorie") extended Kummer's theory to the case of general algebraic integers.

§3. Halter-Koch's axiomatic approach to a general version of the Kronecker function ring and spaces of valuation domains

Toward the middle of the XIXth century, E.E. Kummer discovered that the ring of integers of a cyclotomic field does not have the unique factorization property.

Few years later, in 1847 Kummer introduced the concept of "ideal numbers" to re-establish some of the factorization theory for cyclotomic integers with prime exponents. (In 1856 he generalized his theory to the case of cyclotomic integers with arbitrary exponents.)

R. Dedekind in 1871 (XI supplement to Dirichlet's "Vorlesungen über Zahlentheorie") extended Kummer's theory to the case of general algebraic integers.

§3. Halter-Koch's axiomatic approach to a general version of the Kronecker function ring and spaces of valuation domains

Toward the middle of the XIXth century, E.E. Kummer discovered that the ring of integers of a cyclotomic field does not have the unique factorization property.

Few years later, in 1847 Kummer introduced the concept of "ideal numbers" to re-establish some of the factorization theory for cyclotomic integers with prime exponents. (In 1856 he generalized his theory to the case of cyclotomic integers with arbitrary exponents.)

R. Dedekind in 1871 (XI supplement to Dirichlet's "Vorlesungen über Zahlentheorie") extended Kummer's theory to the case of general algebraic integers.

L. Kronecker has essentially achieved a similar goal in 1859, about 12 years after Kummer's pioneering work, but he published nothing until 1882 (the paper appeared in honor of the 50th anniversary of Kummer's doctorate).

Kronecker's theory holds in a larger context than that of ring of integers of algebraic numbers and solves a more general problem.

The primary objective of his theory was to extend the concept of divisibility in such a way any finite set of elements has a GCD (greatest common divisor).

****

Main references for the "classical" Kronecker function ring

L. Kronecker (1882), W. Krull (1936), H. Weyl (1940), H.M. Edwards (1990).

L. Kronecker has essentially achieved a similar goal in 1859, about 12 years after Kummer's pioneering work, but he published nothing until 1882 (the paper appeared in honor of the 50th anniversary of Kummer's doctorate).

Kronecker's theory holds in a larger context than that of ring of integers of algebraic numbers and solves a more general problem.

The primary objective of his theory was to extend the concept of divisibility in such a way any finite set of elements has a GCD (greatest common divisor).

****

Main references for the "classical" Kronecker function ring

L. Kronecker (1882), W. Krull (1936), H. Weyl (1940), H.M. Edwards (1990).

L. Kronecker has essentially achieved a similar goal in 1859, about 12 years after Kummer's pioneering work, but he published nothing until 1882 (the paper appeared in honor of the 50th anniversary of Kummer's doctorate).

Kronecker's theory holds in a larger context than that of ring of integers of algebraic numbers and solves a more general problem.

The primary objective of his theory was to extend the concept of divisibility in such a way any finite set of elements has a GCD (greatest common divisor).

****

Main references for the "classical" Kronecker function ring

L. Kronecker (1882), W. Krull (1936), H. Weyl (1940), H.M. Edwards (1990).

L. Kronecker has essentially achieved a similar goal in 1859, about 12 years after Kummer's pioneering work, but he published nothing until 1882 (the paper appeared in honor of the 50th anniversary of Kummer's doctorate).

Kronecker's theory holds in a larger context than that of ring of integers of algebraic numbers and solves a more general problem.

The primary objective of his theory was to extend the concept of divisibility in such a way any finite set of elements has a GCD (greatest common divisor).

****

Main references for the "classical" Kronecker function ring

L. Kronecker (1882), W. Krull (1936), H. Weyl (1940), H.M. Edwards (1990).

With a modern terminology and notation, *the Kronecker function ring of a Dedekind domain D* is given by:

$$\begin{aligned}\mathrm{Kr}(D) &:= \left\{ \frac{f}{g} \mid f, g \in D[X] \ \textit{and} \ \mathbf{c}(f) \subseteq \mathbf{c}(g) \right\} \\ &= \left\{ \frac{f'}{g'} \mid f', g' \in D[X] \ \textit{and} \ \mathbf{c}(g') = D \right\},\end{aligned}$$

(where $\mathbf{c}(h)$ denotes *the content* of a polynomial $h \in D[X]$, i.e. the ideal of D generated by the coefficients of h).

Note that the previous equality holds since we are assuming that D is a Dedekind domain (e.g., the integral closure of a PID D_0 in a finite field extension K of the quotient field K_0 of D_0).

In this case, for each polynomial $g \in D[X]$, $\mathbf{c}(g)$ is an invertible ideal of D and, by choosing a polynomial $u \in K[X]$ such that $\mathbf{c}(u) = (\mathbf{c}(g))^{-1}$, then we have $f/g = uf/ug = f'/g'$, with $f' := uf$, $g' := ug \in D[X]$ and, obviously, $\mathbf{c}(g') = D$.

With a modern terminology and notation, *the Kronecker function ring of a Dedekind domain D* is given by:

$$\begin{aligned}\mathtt{Kr}(D) &:= \left\{\frac{f}{g} \mid f, g \in D[X] \ \textit{and} \ \mathbf{c}(f) \subseteq \mathbf{c}(g)\right\} \\ &= \left\{\frac{f'}{g'} \mid f', g' \in D[X] \ \textit{and} \ \mathbf{c}(g') = D\right\},\end{aligned}$$

(where $\mathbf{c}(h)$ denotes *the content* of a polynomial $h \in D[X]$, i.e. the ideal of D generated by the coefficients of h).

Note that the previous equality holds since we are assuming that D is a Dedekind domain (e.g., the integral closure of a PID D_0 in a finite field extension K of the quotient field K_0 of D_0).

In this case, for each polynomial $g \in D[X]$, $\mathbf{c}(g)$ is an invertible ideal of D and, by choosing a polynomial $u \in K[X]$ such that $\mathbf{c}(u) = (\mathbf{c}(g))^{-1}$, then we have $f/g = uf/ug = f'/g'$, with $f' := uf$, $g' := ug \in D[X]$ and, obviously, $\mathbf{c}(g') = D$.

With a modern terminology and notation, *the Kronecker function ring of a Dedekind domain D* is given by:

$$\begin{aligned}\mathtt{Kr}(D) &:= \left\{\frac{f}{g} \mid f, g \in D[X] \;\; and \;\; \mathbf{c}(f) \subseteq \mathbf{c}(g)\right\} \\ &= \left\{\frac{f'}{g'} \mid f', g' \in D[X] \;\; and \;\; \mathbf{c}(g') = D\right\},\end{aligned}$$

(where $\mathbf{c}(h)$ denotes *the content* of a polynomial $h \in D[X]$, i.e. the ideal of D generated by the coefficients of h).

Note that the previous equality holds since we are assuming that D is a Dedekind domain (e.g., the integral closure of a PID D_0 in a finite field extension K of the quotient field K_0 of D_0).

In this case, for each polynomial $g \in D[X]$, $\mathbf{c}(g)$ is an invertible ideal of D and, by choosing a polynomial $u \in K[X]$ such that $\mathbf{c}(u) = (\mathbf{c}(g))^{-1}$, then we have $f/g = uf/ug = f'/g'$, with $f' := uf$, $g' := ug \in D[X]$ and, obviously, $\mathbf{c}(g') = D$.

With a modern terminology and notation, *the Kronecker function ring of a Dedekind domain D* is given by:

$$\begin{aligned}\mathtt{Kr}(D) :=& \left\{\frac{f}{g} \mid f, g \in D[X] \;\; and \;\; \mathbf{c}(f) \subseteq \mathbf{c}(g)\right\} \\ =& \left\{\frac{f'}{g'} \mid f', g' \in D[X] \;\; and \;\; \mathbf{c}(g') = D\right\},\end{aligned}$$

(where $\mathbf{c}(h)$ denotes *the content* of a polynomial $h \in D[X]$, i.e. the ideal of D generated by the coefficients of h).

Note that the previous equality holds since we are assuming that D is a Dedekind domain (e.g., the integral closure of a PID D_0 in a finite field extension K of the quotient field K_0 of D_0).

In this case, for each polynomial $g \in D[X]$, $\mathbf{c}(g)$ is an invertible ideal of D and, by choosing a polynomial $u \in K[X]$ such that $\mathbf{c}(u) = (\mathbf{c}(g))^{-1}$, then we have $f/g = uf/ug = f'/g'$, with $f' := uf$, $g' := ug \in D[X]$ and, obviously, $\mathbf{c}(g') = D$.

The fundamental properties of the Kronecker function ring are the following:

(1) *$\mathrm{Kr}(D)$ is a Bézout domain (i.e. each finite set of elements has a GCD and the GCD can be expressed as linear combination of these elements) and $D[X] \subseteq \mathrm{Kr}(D) \subseteq K(X)$ (in particular, the field of rational functions $K(X)$ is the quotient field of $\mathrm{Kr}(D)$).*

(2) *Let $a_0, a_1, \ldots, a_n \in D$ and set $f := a_0 + a_1X + \ldots + a_nX^n \in D[X]$, then:*

$$(a_0, a_1, \ldots, a_n)\mathrm{Kr}(D) = f\mathrm{Kr}(D) \ \ (\textit{thus, } GCD_{\mathrm{Kr}(D)}(a_0, a_1, \ldots, a_n) = f),$$
$$f\mathrm{Kr}(D) \cap K = (a_0, a_1, \ldots, a_n)D = \mathbf{c}(f)D \ \ (\textit{hence, } \mathrm{Kr}(D) \cap K = D).$$

The fundamental properties of the Kronecker function ring are the following:

(1) *$\mathtt{Kr}(D)$ is a Bézout domain (i.e. each finite set of elements has a GCD and the GCD can be expressed as linear combination of these elements) and $D[X] \subseteq \mathtt{Kr}(D) \subseteq K(X)$ (in particular, the field of rational functions $K(X)$ is the quotient field of $\mathtt{Kr}(D)$).*

(2) *Let $a_0, a_1, \ldots, a_n \in D$ and set $f := a_0 + a_1X + \ldots + a_nX^n \in D[X]$, then:*

$$(a_0, a_1, \ldots, a_n)\mathtt{Kr}(D) = f\mathtt{Kr}(D) \ \ (\textit{thus, } GCD_{\mathtt{Kr}(D)}(a_0, a_1, \ldots, a_n) = f),$$
$$f\mathtt{Kr}(D) \cap K = (a_0, a_1, \ldots, a_n)D = \mathbf{c}(f)D \ \ (\textit{hence, } \mathtt{Kr}(D) \cap K = D).$$

The fundamental properties of the Kronecker function ring are the following:

(1) *$\mathrm{Kr}(D)$ is a Bézout domain (i.e. each finite set of elements has a GCD and the GCD can be expressed as linear combination of these elements) and $D[X] \subseteq \mathrm{Kr}(D) \subseteq K(X)$ (in particular, the field of rational functions $K(X)$ is the quotient field of $\mathrm{Kr}(D)$).*

(2) *Let $a_0, a_1, \ldots, a_n \in D$ and set $f := a_0 + a_1X + \ldots + a_nX^n \in D[X]$, then:*

$$(a_0, a_1, \ldots, a_n)\mathrm{Kr}(D) = f\mathrm{Kr}(D) \ (\textit{thus, } GCD_{\mathrm{Kr}(D)}(a_0, a_1, \ldots, a_n) = f),$$
$$f\mathrm{Kr}(D) \cap K = (a_0, a_1, \ldots, a_n)D = \mathbf{c}(f)D \ \ (\textit{hence, } \mathrm{Kr}(D) \cap K = D).$$

Kronecker function rings play a special role in the investigation of spaces of valuation domains.

The motivations for studying, from a topological point of view, spaces of valuation domains come from various directions and, historically, mainly

• from Zariski's work for the reduction of singularities of an algebraic surface and, more generally, for establishing new foundations of algebraic geometry by algebraic means
(see [Zariski, 1939], [Zariski, 1944] and [Zariski-Samuel, 1960]).

Kronecker function rings play a special role in the investigation of spaces of valuation domains.

The motivations for studying, from a topological point of view, spaces of valuation domains come from various directions and, historically, mainly

• from Zariski's work for the reduction of singularities of an algebraic surface and, more generally, for establishing new foundations of algebraic geometry by algebraic means
(see [Zariski, 1939], [Zariski, 1944] and [Zariski-Samuel, 1960]).

Kronecker function rings play a special role in the investigation of spaces of valuation domains.

The motivations for studying, from a topological point of view, spaces of valuation domains come from various directions and, historically, mainly

• from Zariski's work for the reduction of singularities of an algebraic surface and, more generally, for establishing new foundations of algebraic geometry by algebraic means
(see [Zariski, 1939], [Zariski, 1944] and [Zariski-Samuel, 1960]).

Kronecker function rings play a special role in the investigation of spaces of valuation domains.

The motivations for studying, from a topological point of view, spaces of valuation domains come from various directions and, historically, mainly

• from Zariski's work for the reduction of singularities of an algebraic surface and, more generally, for establishing new foundations of algebraic geometry by algebraic means
(see [Zariski, 1939], [Zariski, 1944] and [Zariski-Samuel, 1960]).

NOTATION

- Let K be a field and A a subring (*possibly, a subfield*) of K
- Let

$$\mathtt{Zar}(K|A) := \{V \mid V \text{ valuation domain with } A \subseteq V \subseteq K = \mathrm{qf}(V)\}\,.$$

- In case A is the prime subring of K, then $\mathtt{Zar}(K|A)$ includes all valuation domains with K as quotient field and we denote it by simply $\mathtt{Zar}(K)$.

- In case A is an integral domain with quotient field K, $A \neq K$, then $\mathtt{Zar}(K|A)$ is the set of all valuation overrings of A and we simply denote it by $\mathtt{Zar}(A)$.

NOTATION

- Let K be a field and A a subring (*possibly, a subfield*) of K
- Let

$$\mathtt{Zar}(K|A) := \{V \mid V \text{ valuation domain with } A \subseteq V \subseteq K = \mathtt{qf}(V)\}.$$

- In case A is the prime subring of K, then $\mathtt{Zar}(K|A)$ includes all valuation domains with K as quotient field and we denote it by simply $\mathtt{Zar}(K)$.
- In case A is an integral domain with quotient field K, $A \neq K$, then $\mathtt{Zar}(K|A)$ is the set of all valuation overrings of A and we simply denote it by $\mathtt{Zar}(A)$.

NOTATION

- Let K be a field and A a subring (*possibly, a subfield*) of K
- Let

$$\mathtt{Zar}(K|A) := \{V \mid V \text{ valuation domain with } A \subseteq V \subseteq K = \mathrm{qf}(V)\}.$$

- In case A is the prime subring of K, then $\mathtt{Zar}(K|A)$ includes all valuation domains with K as quotient field and we denote it by simply $\mathtt{Zar}(K)$.

- In case A is an integral domain with quotient field K, $A \neq K$, then $\mathtt{Zar}(K|A)$ is the set of all valuation overrings of A and we simply denote it by $\mathtt{Zar}(A)$.

NOTATION

- Let K be a field and A a subring (*possibly, a subfield*) of K
- Let

$$\mathtt{Zar}(K|A) := \{V \mid V \text{ valuation domain with } A \subseteq V \subseteq K = \mathtt{qf}(V)\}\,.$$

- In case A is the prime subring of K, then $\mathtt{Zar}(K|A)$ includes all valuation domains with K as quotient field and we denote it by simply $\mathtt{Zar}(K)$.

- In case A is an integral domain with quotient field K, $A \neq K$, then $\mathtt{Zar}(K|A)$ is the set of all valuation overrings of A and we simply denote it by $\mathtt{Zar}(A)$.

- A first topological approach to the space $\mathtt{Zar}(K|A)$ is due to O. Zariski who proved *the quasi-compactness of this space, endowed with* what is now called *Zariski topology* (see [Zariski, 1944] and [Zariski-Samuel, 1960]).

The topological structure on $Z := \mathtt{Zar}(K|A)$ is defined by taking, as *a basis for the open sets*, the subsets $\mathcal{U}_F := \{V \in Z \mid V \supseteq F\}$ for F varying in the finite subsets of K, i.e., if $F := \{x_1, x_2, \ldots, x_n\}$, with $x_i \in K$, then

$$\mathcal{U}_F = \mathtt{Zar}(K|A[x_1, x_2, \ldots, x_n]).$$

- The space $Z = \mathtt{Zar}(K|A)$, equipped with this topology, is usually called *the Zariski-Riemann space* (or, sometimes, *the abstract Zariski-Riemann surface*) of K over A.

- Note that recently B. Olberding, 2014 has introduced and studied also a natural structure of locally ringed space on $Z = \mathtt{Zar}(K|A)$, which realizes the Zariski-Riemann space as a projective limit of projective integral schemes over $\mathrm{Spec}(A)$ whose function field is a subfield of K.

- A first topological approach to the space $\mathtt{Zar}(K|A)$ is due to O. Zariski who proved *the quasi-compactness of this space, endowed with* what is now called *Zariski topology*
(see [Zariski, 1944] and [Zariski-Samuel, 1960]).

The topological structure on $Z := \mathtt{Zar}(K|A)$ is defined by taking, as *a basis for the open sets*, the subsets $\mathcal{U}_F := \{V \in Z \mid V \supseteq F\}$ for F varying in the finite subsets of K, i.e., if $F := \{x_1, x_2, \ldots, x_n\}$, with $x_i \in K$, then

$$\mathcal{U}_F = \mathtt{Zar}(K|A[x_1, x_2, \ldots, x_n]).$$

- The space $Z = \mathtt{Zar}(K|A)$, equipped with this topology, is usually called *the Zariski-Riemann space* (or, sometimes, *the abstract Zariski-Riemann surface*) of K over A.

- Note that recently B. Olberding, 2014 has introduced and studied also a natural structure of locally ringed space on $Z = \mathtt{Zar}(K|A)$, which realizes the Zariski-Riemann space as a projective limit of projective integral schemes over $\mathrm{Spec}(A)$ whose function field is a subfield of K.

- A first topological approach to the space $\mathtt{Zar}(K|A)$ is due to O. Zariski who proved *the quasi-compactness of this space, endowed with what is now called Zariski topology*
(see [Zariski, 1944] and [Zariski-Samuel, 1960]).

The topological structure on $Z := \mathtt{Zar}(K|A)$ is defined by taking, as *a basis for the open sets*, the subsets $\mathcal{U}_F := \{V \in Z \mid V \supseteq F\}$ for F varying in the finite subsets of K, i.e., if $F := \{x_1, x_2, \dots, x_n\}$, with $x_i \in K$, then

$$\mathcal{U}_F = \mathtt{Zar}(K|A[x_1, x_2, \dots, x_n]).$$

- The space $Z = \mathtt{Zar}(K|A)$, equipped with this topology, is usually called *the Zariski-Riemann space* (or, sometimes, *the abstract Zariski-Riemann surface*) of K over A.

- Note that recently B. Olberding, 2014 has introduced and studied also a natural structure of locally ringed space on $Z = \mathtt{Zar}(K|A)$, which realizes the Zariski-Riemann space as a projective limit of projective integral schemes over $\mathrm{Spec}(A)$ whose function field is a subfield of K.

- A first topological approach to the space $\mathtt{Zar}(K|A)$ is due to O. Zariski who proved *the quasi-compactness of this space, endowed with* what is now called *Zariski topology* (see [Zariski, 1944] and [Zariski-Samuel, 1960]).

The topological structure on $Z := \mathtt{Zar}(K|A)$ is defined by taking, as *a basis for the open sets*, the subsets $\mathcal{U}_F := \{V \in Z \mid V \supseteq F\}$ for F varying in the finite subsets of K, i.e., if $F := \{x_1, x_2, \dots, x_n\}$, with $x_i \in K$, then

$$\mathcal{U}_F = \mathtt{Zar}(K|A[x_1, x_2, \dots, x_n]).$$

- The space $Z = \mathtt{Zar}(K|A)$, equipped with this topology, is usually called *the Zariski-Riemann space* (or, sometimes, *the abstract Zariski-Riemann surface*) of K over A.

- Note that recently B. Olberding, 2014 has introduced and studied also a natural structure of locally ringed space on $Z = \mathtt{Zar}(K|A)$, which realizes the Zariski-Riemann space as a projective limit of projective integral schemes over $\mathrm{Spec}(A)$ whose function field is a subfield of K.

- A first topological approach to the space $\mathtt{Zar}(K|A)$ is due to O. Zariski who proved *the quasi-compactness of this space, endowed with* what is now called *Zariski topology*
(see [Zariski, 1944] and [Zariski-Samuel, 1960]).

The topological structure on $Z := \mathtt{Zar}(K|A)$ is defined by taking, as *a basis for the open sets*, the subsets $\mathcal{U}_F := \{V \in Z \mid V \supseteq F\}$ for F varying in the finite subsets of K, i.e., if $F := \{x_1, x_2, \ldots, x_n\}$, with $x_i \in K$, then

$$\mathcal{U}_F = \mathtt{Zar}(K|A[x_1, x_2, \ldots, x_n]).$$

- The space $Z = \mathtt{Zar}(K|A)$, equipped with this topology, is usually called *the Zariski-Riemann space* (or, sometimes, *the abstract Zariski-Riemann surface*) of K over A.

- Note that recently B. Olberding, 2014 has introduced and studied also a natural structure of locally ringed space on $Z = \mathtt{Zar}(K|A)$, which realizes the Zariski-Riemann space as a projective limit of projective integral schemes over $\mathtt{Spec}(A)$ whose function field is a subfield of K.

From the collaboration with David Dobbs in 1986/87, we published two papers (one of these, joint also with Rich Fedder) concerning the topological and algebraic structure of the space $\mathtt{Zar}(A)$.
(see [Dobbs-Fedder-Fontana, 1987], [Dobbs-Fontana, 1986]).

- First we proved, using a purely topological approach that: *If K is the quotient field of A then $\mathtt{Zar}(A)$, endowed with the Zariski topology, is a spectral space* in the sense of [Hochster, 1969] (see [Dobbs-Fedder-Fontana, 1987]).

This result was later proved by several authors with a variety of different techniques:

- in [Kuhlmann, 2004, Appendix], using a model-theoretic approach;
- in [Finocchiaro, 2013, Corollary 3.3] using new topological methods;
- in [Schwartz, 2013], using the inverse spectrum of a lattice ordered abelian group (via the Jaffard-Ohm Theorem).

From the collaboration with David Dobbs in 1986/87, we published two papers (one of these, joint also with Rich Fedder) concerning the topological and algebraic structure of the space $\mathtt{Zar}(A)$.
(see [Dobbs-Fedder-Fontana, 1987], [Dobbs-Fontana, 1986]).

- First we proved, using a purely topological approach that:
If K is the quotient field of A then $\mathtt{Zar}(A)$, endowed with the Zariski topology, is a spectral space in the sense of [Hochster, 1969]
(see [Dobbs-Fedder-Fontana, 1987]).

This result was later proved by several authors with a variety of different techniques:

- in [Kuhlmann, 2004, Appendix], using a model-theoretic approach;
- in [Finocchiaro, 2013, Corollary 3.3] using new topological methods;
- in [Schwartz, 2013], using the inverse spectrum of a lattice ordered abelian group (via the Jaffard-Ohm Theorem).

From the collaboration with David Dobbs in 1986/87, we published two papers (one of these, joint also with Rich Fedder) concerning the topological and algebraic structure of the space $\mathtt{Zar}(A)$.
(see [Dobbs-Fedder-Fontana, 1987], [Dobbs-Fontana, 1986]).

- First we proved, using a purely topological approach that:
If K is the quotient field of A then $\mathtt{Zar}(A)$, endowed with the Zariski topology, is a spectral space in the sense of [Hochster, 1969]
(see [Dobbs-Fedder-Fontana, 1987]).

This result was later proved by several authors with a variety of different techniques:

- in [Kuhlmann, 2004, Appendix], using a model-theoretic approach;
- in [Finocchiaro, 2013, Corollary 3.3] using new topological methods;
- in [Schwartz, 2013], using the inverse spectrum of a lattice ordered abelian group (via the Jaffard-Ohm Theorem).

From the collaboration with David Dobbs in 1986/87, we published two papers (one of these, joint also with Rich Fedder) concerning the topological and algebraic structure of the space $\mathtt{Zar}(A)$.
(see [Dobbs-Fedder-Fontana, 1987], [Dobbs-Fontana, 1986]).

- First we proved, using a purely topological approach that:
If K is the quotient field of A then $\mathtt{Zar}(A)$, endowed with the Zariski topology, is a spectral space in the sense of [Hochster, 1969]
(see [Dobbs-Fedder-Fontana, 1987]).

This result was later proved by several authors with a variety of different techniques:

- in [Kuhlmann, 2004, Appendix], using a model-theoretic approach;
- in [Finocchiaro, 2013, Corollary 3.3] using new topological methods;
- in [Schwartz, 2013], using the inverse spectrum of a lattice ordered abelian group (via the Jaffard-Ohm Theorem).

From the collaboration with David Dobbs in 1986/87, we published two papers (one of these, joint also with Rich Fedder) concerning the topological and algebraic structure of the space $\mathtt{Zar}(A)$.
(see [Dobbs-Fedder-Fontana, 1987], [Dobbs-Fontana, 1986]).

• First we proved, using a purely topological approach that:
If K is the quotient field of A then $\mathtt{Zar}(A)$, endowed with the Zariski topology, is a spectral space in the sense of [Hochster, 1969]
(see [Dobbs-Fedder-Fontana, 1987]).

This result was later proved by several authors with a variety of different techniques:

- in [Kuhlmann, 2004, Appendix], using a model-theoretic approach;
- in [Finocchiaro, 2013, Corollary 3.3] using new topological methods;
- in [Schwartz, 2013], using the inverse spectrum of a lattice ordered abelian group (via the Jaffard-Ohm Theorem).

- Immediately after the first paper, in collaboration with David Dobbs, we proved a more precise result, exhibiting explicitly an integral domain $\mathcal{A}$ with a canonical map $\varphi : \mathtt{Zar}(A) \to \mathtt{Spec}(\mathcal{A})$ realizing a topological homeomorphism (with respect to the Zariski topologies).

We need some preliminaries.

Let A be an integral domain with quotient field K and let $\overline{A}$ be the integral closure of A, extending Kronecker's classical theory (concerning rings of algebraic numbers), in [Krull, 1936] the author introduced on $\overline{A}$ what we call now

- *the Kronecker function ring of $\overline{A}$ with respect to the (star) b–operation* (or, completion), i.e., the integral domain

$$\mathtt{Kr}(\overline{A}, b) := \{f/g \in K(X) \mid \mathbf{c}(f)^b \subseteq \mathbf{c}(g)^b\},$$

where the b–operation is defined, for each nonzero fractional ideal E of $\overline{A}$ by $E^b := \bigcap\{EV \mid V \in \mathtt{Zar}(A) = \mathtt{Zar}(\overline{A})\}$.

- Immediately after the first paper, in collaboration with David Dobbs, we proved a more precise result, exhibiting explicitly an integral domain $\boldsymbol{\mathcal{A}}$ with a canonical map $\varphi : \mathtt{Zar}(A) \to \mathtt{Spec}(\boldsymbol{\mathcal{A}})$ realizing a topological homeomorphism (with respect to the Zariski topologies).

We need some preliminaries.

Let A be an integral domain with quotient field K and let $\overline{A}$ be the integral closure of A, extending Kronecker's classical theory (concerning rings of algebraic numbers), in [Krull, 1936] the author introduced on $\overline{A}$ what we call now

- *the Kronecker function ring of $\overline{A}$ with respect to the (star) b–operation* (or, completion), i.e., the integral domain

$$\mathrm{Kr}(\overline{A}, b) := \{f/g \in K(X) \mid \mathbf{c}(f)^b \subseteq \mathbf{c}(g)^b\},$$

where the b–operation is defined, for each nonzero fractional ideal E of $\overline{A}$ by $E^b := \bigcap\{EV \mid V \in \mathtt{Zar}(A) = \mathtt{Zar}(\overline{A})\}$.

- Immediately after the first paper, in collaboration with David Dobbs, we proved a more precise result, exhibiting explicitly an integral domain $\boldsymbol{\mathcal{A}}$ with a canonical map $\varphi : \mathtt{Zar}(A) \to \mathtt{Spec}(\boldsymbol{\mathcal{A}})$ realizing a topological homeomorphism (with respect to the Zariski topologies).

We need some preliminaries.

Let A be an integral domain with quotient field K and let $\overline{A}$ be the integral closure of A, extending Kronecker's classical theory (concerning rings of algebraic numbers), in [Krull, 1936] the author introduced on $\overline{A}$ what we call now

- *the Kronecker function ring of $\overline{A}$ with respect to the (star) b–operation* (or, completion), i.e., the integral domain

$$\mathtt{Kr}(\overline{A}, b) := \{f/g \in K(X) \mid \mathbf{c}(f)^b \subseteq \mathbf{c}(g)^b\},$$

where the b–operation is defined, for each nonzero fractional ideal E of $\overline{A}$ by $E^b := \bigcap\{EV \mid V \in \mathtt{Zar}(A) = \mathtt{Zar}(\overline{A})\}$.

- Immediately after the first paper, in collaboration with David Dobbs, we proved a more precise result, exhibiting explicitly an integral domain $\boldsymbol{\mathcal{A}}$ with a canonical map $\varphi : \mathtt{Zar}(A) \to \mathtt{Spec}(\boldsymbol{\mathcal{A}})$ realizing a topological homeomorphism (with respect to the Zariski topologies).

We need some preliminaries.

Let A be an integral domain with quotient field K and let $\overline{A}$ be the integral closure of A, extending Kronecker's classical theory (concerning rings of algebraic numbers), in [Krull, 1936] the author introduced on $\overline{A}$ what we call now

- *the Kronecker function ring of $\overline{A}$ with respect to the (star) b–operation* (or, completion), i.e., the integral domain

$$\mathtt{Kr}(\overline{A}, b) := \{f/g \in K(X) \mid \mathbf{c}(f)^b \subseteq \mathbf{c}(g)^b\},$$

where the b–operation is defined, for each nonzero fractional ideal E of $\overline{A}$ by $E^b := \bigcap\{EV \mid V \in \mathtt{Zar}(A) = \mathtt{Zar}(\overline{A})\}$.

• Immediately after the first paper, in collaboration with David Dobbs, we proved a more precise result, exhibiting explicitly an integral domain $\boldsymbol{\mathcal{A}}$ with a canonical map $\varphi : \mathtt{Zar}(A) \to \mathtt{Spec}(\boldsymbol{\mathcal{A}})$ realizing a topological homeomorphism (with respect to the Zariski topologies).

We need some preliminaries.

Let A be an integral domain with quotient field K and let $\overline{A}$ be the integral closure of A, extending Kronecker's classical theory (concerning rings of algebraic numbers), in [Krull, 1936] the author introduced on $\overline{A}$ what we call now

• *the Kronecker function ring of $\overline{A}$ with respect to the (star) b–operation* (or, completion), i.e., the integral domain

$$\mathtt{Kr}(\overline{A}, b) := \{f/g \in K(X) \mid \mathbf{c}(f)^b \subseteq \mathbf{c}(g)^b\},$$

where the b–operation is defined, for each nonzero fractional ideal E of $\overline{A}$ by $E^b := \bigcap\{EV \mid V \in \mathtt{Zar}(A) = \mathtt{Zar}(\overline{A})\}$.

Using the previous notation, we have

Theorem 1, [Dobbs-Fontana, 1986]

Let A be an integral domain with quotient field K, and let $\mathcal{A} := \mathtt{Kr}(\overline{A}, b)$. The canonical map

$$\varphi : \mathtt{Zar}(A) \to \mathtt{Spec}(\mathcal{A}), \ (V, M) \ \mapsto \ M(X) \cap \mathcal{A}$$

is a homeomorphism (with respect to the Zariski topologies).

Note that this theorem did not concern the more general space $\mathtt{Zar}(K|A)$.

A result including the case of $\mathtt{Zar}(K|A)$ was possible many years later, only after appropriate generalizations of the Kronecker function ring were introduced and studied.

Using the previous notation, we have

Theorem 1, [Dobbs-Fontana, 1986]

Let A be an integral domain with quotient field K, and let $\boldsymbol{\mathcal{A}} := \mathtt{Kr}(\overline{A}, b)$. The canonical map

$$\varphi : \mathtt{Zar}(A) \to \mathtt{Spec}(\boldsymbol{\mathcal{A}}), \ (V, M) \ \mapsto \ M(X) \cap \boldsymbol{\mathcal{A}}$$

is a homeomorphism (with respect to the Zariski topologies).

Note that this theorem did not concern the more general space $\mathtt{Zar}(K|A)$.

A result including the case of $\mathtt{Zar}(K|A)$ was possible many years later, only after appropriate generalizations of the Kronecker function ring were introduced and studied.

Using the previous notation, we have

Theorem 1, [Dobbs-Fontana, 1986]

Let A be an integral domain with quotient field K, and let $\boldsymbol{\mathcal{A}} := \mathtt{Kr}(\overline{A}, b)$. The canonical map

$$\varphi : \mathtt{Zar}(A) \to \mathtt{Spec}(\boldsymbol{\mathcal{A}}) \,, \; (V, M) \;\mapsto\; M(X) \cap \boldsymbol{\mathcal{A}}$$

is a homeomorphism (with respect to the Zariski topologies).

Note that this theorem did not concern the more general space $\mathtt{Zar}(K|A)$.

A result including the case of $\mathtt{Zar}(K|A)$ was possible many years later, only after appropriate generalizations of the Kronecker function ring were introduced and studied.

Using the previous notation, we have

Theorem 1, [Dobbs-Fontana, 1986]

Let A be an integral domain with quotient field K, and let $\boldsymbol{\mathcal{A}} := \mathtt{Kr}(\overline{A}, b)$. The canonical map

$$\varphi : \mathtt{Zar}(A) \to \mathtt{Spec}(\boldsymbol{\mathcal{A}}) \,,\; (V, M) \;\mapsto\; M(X) \cap \boldsymbol{\mathcal{A}}$$

is a homeomorphism (with respect to the Zariski topologies).

Note that this theorem did not concern the more general space $\mathtt{Zar}(K|A)$.

A result including the case of $\mathtt{Zar}(K|A)$ was possible many years later, only after appropriate generalizations of the Kronecker function ring were introduced and studied.

- A first generalization of the Kronecker function ring *for any integral domain* (not necessarily integrally closed) and *for any semistar operation* (non necessarily e.a.b., as the *b*-operation is) was given and studied by Fontana-Loper in two papers published in 2001 and 2003.

- Another generalization, based on an axiomatic approach, was given in [Halter-Koch, 2003].

More precisely, Halter-Koch gives the following "abstract" definition:

Let K be a field, X an indeterminate over K, R a subring of $K(X)$ and $A := R \cap K$. If

- $X \in \mathbf{U}(R)$ (i.e. X is a unit in R);
- $f(0) \in f \cdot R$ for each $f \in K[X]$;

then R is called *a K–function ring of A*.

- A first generalization of the Kronecker function ring *for any integral domain* (not necessarily integrally closed) and *for any semistar operation* (non necessarily e.a.b., as the b-operation is) was given and studied by Fontana-Loper in two papers published in 2001 and 2003.

- Another generalization, based on an axiomatic approach, was given in [Halter-Koch, 2003].

More precisely, Halter-Koch gives the following "abstract" definition:

Let K be a field, X an indeterminate over K, R a subring of $K(X)$ and $A := R \cap K$. If

- $X \in \mathbf{U}(R)$ (i.e. X is a unit in R);
- $f(0) \in f \cdot R$ for each $f \in K[X]$;

then R is called *a K–function ring of A*.

- A first generalization of the Kronecker function ring *for any integral domain* (not necessarily integrally closed) and *for any semistar operation* (non necessarily e.a.b., as the b-operation is) was given and studied by Fontana-Loper in two papers published in 2001 and 2003.

- Another generalization, based on an axiomatic approach, was given in [Halter-Koch, 2003].

More precisely, Halter-Koch gives the following "abstract" definition:

Let K be a field, X an indeterminate over K, R a subring of $K(X)$ and $A := R \cap K$. If

- $X \in \mathbf{U}(R)$ (i.e. X is a unit in R);
- $f(0) \in f \cdot R$ for each $f \in K[X]$;

then R is called *a K–function ring of A.*

- A first generalization of the Kronecker function ring *for any integral domain* (not necessarily integrally closed) and *for any semistar operation* (non necessarily e.a.b., as the *b*-operation is) was given and studied by Fontana-Loper in two papers published in 2001 and 2003.

- Another generalization, based on an axiomatic approach, was given in [Halter-Koch, 2003].

More precisely, Halter-Koch gives the following "abstract" definition:

Let K be a field, X an indeterminate over K, R a subring of $K(X)$ and $A := R \cap K$. If

- $X \in \mathbf{U}(R)$ (i.e. X is a unit in R);
- $f(0) \in f \cdot R$ for each $f \in K[X]$;

then R is called *a K–function ring of A.*

- A first generalization of the Kronecker function ring *for any integral domain* (not necessarily integrally closed) and *for any semistar operation* (non necessarily e.a.b., as the b-operation is) was given and studied by Fontana-Loper in two papers published in 2001 and 2003.

- Another generalization, based on an axiomatic approach, was given in [Halter-Koch, 2003].

More precisely, Halter-Koch gives the following "abstract" definition:

Let K be a field, X an indeterminate over K, R a subring of $K(X)$ and $A := R \cap K$. If

- $X \in \mathbf{U}(R)$ (i.e. X is a unit in R);
- $f(0) \in f \cdot R$ for each $f \in K[X]$;

then R is called *a K–function ring of A.*

- A first generalization of the Kronecker function ring *for any integral domain* (not necessarily integrally closed) and *for any semistar operation* (non necessarily e.a.b., as the b-operation is) was given and studied by Fontana-Loper in two papers published in 2001 and 2003.

- Another generalization, based on an axiomatic approach, was given in [Halter-Koch, 2003].

More precisely, Halter-Koch gives the following "abstract" definition:

Let K be a field, X an indeterminate over K, R a subring of $K(X)$ and $A := R \cap K$. If

- $X \in \mathbf{U}(R)$ (i.e. X is a unit in R);
- $f(0) \in f \cdot R$ for each $f \in K[X]$;

then R is called *a K–function ring of A*.

- A first generalization of the Kronecker function ring *for any integral domain* (not necessarily integrally closed) and *for any semistar operation* (non necessarily e.a.b., as the b-operation is) was given and studied by Fontana-Loper in two papers published in 2001 and 2003.

- Another generalization, based on an axiomatic approach, was given in [Halter-Koch, 2003].

More precisely, Halter-Koch gives the following "abstract" definition:

Let K be a field, X an indeterminate over K, R a subring of $K(X)$ and $A := R \cap K$. If

- $X \in \mathbf{U}(R)$ (i.e. X is a unit in R);
- $f(0) \in f \cdot R$ for each $f \in K[X]$;

then R is called *a K–function ring of A*.

Using only these two axioms, he proved that *R (in $K(X)$) "behaves as a classical Kronecker function ring" for A*, i.e.,

Theorem, Halter-Koch 2003

Let R be a K-function ring of $A = R \cap K$, then:

- *R is a Bézout domain with quotient field $K(X)$.*
- *A is integrally closed in K.*
- *For each $f := a_0 + a_1X + \cdots + a_nX_n \in K[X]$, then $(a_0, a_1, \ldots, a_n)R = fR$.*

Using only these two axioms, he proved that *R (in $K(X)$) "behaves as a classical Kronecker function ring" for A*, i.e.,

Theorem, Halter-Koch 2003

Let R be a K-function ring of $A = R \cap K$, then:

- *R is a Bézout domain with quotient field $K(X)$.*
- *A is integrally closed in K.*
- *For each $f := a_0 + a_1X + \cdots + a_nX_n \in K[X]$, then $(a_0, a_1, \ldots, a_n)R = fR$.*

Using only these two axioms, he proved that *R (in $K(X)$) "behaves as a classical Kronecker function ring" for A*, i.e.,

Theorem, Halter-Koch 2003

Let R be a K-function ring of $A = R \cap K$, then:

- *R is a Bézout domain with quotient field $K(X)$.*
- *A is integrally closed in K.*
- *For each $f := a_0 + a_1X + \cdots + a_nX_n \in K[X]$, then $(a_0, a_1, \ldots, a_n)R = fR$.*

Using only these two axioms, he proved that *R (in $K(X)$) "behaves as a classical Kronecker function ring" for A*, i.e.,

Theorem, Halter-Koch 2003

Let R be a K-function ring of $A = R \cap K$, then:

- *R is a Bézout domain with quotient field $K(X)$.*
- *A is integrally closed in K.*
- *For each $f := a_0 + a_1X + \cdots + a_nX_n \in K[X]$, then $(a_0, a_1, \ldots, a_n)R = fR$.*

Using only these two axioms, he proved that *R (in $K(X)$) "behaves as a classical Kronecker function ring" for A*, i.e.,

Theorem, Halter-Koch 2003

Let R be a K-function ring of $A = R \cap K$, then:

- *R is a Bézout domain with quotient field $K(X)$.*
- *A is integrally closed in K.*
- *For each $f := a_0 + a_1X + \cdots + a_nX_n \in K[X]$, then $(a_0, a_1, \ldots, a_n)R = fR$.*

Using Halter-Koch's K–function rings, it was proven in [Kwegna-Heubo, 2010] and, independently, in [Finocchiaro-Fontana-Loper, 2013b] as a particular case of a more general result the following:

Theorem 2

Let A be any subring of K, and let

$$\mathtt{Kr}(K|A) := \bigcap \{V(X) \mid V \in \mathtt{Zar}(K|A)\}.$$

Note that $\mathcal{A} := \mathtt{Kr}(K|A)$ is a $K-$function ring, by F. Halter-Koch's theory.

- *The canonical map $\sigma : \mathtt{Zar}(K|A) \to \mathtt{Zar}(K(X)|\mathcal{A})$, $V \mapsto V(X)$ is an homeomorphism.*
- *The canonical map $\varphi : \mathtt{Zar}(K|A) \cong \mathtt{Zar}(K(X)|\mathcal{A}) \to \mathtt{Spec}(\mathcal{A})$ be the map sending a valuation overring of $\mathcal{A}$ into its center on $\mathcal{A}$, composed with the homeomorphism σ, establishes a homeomorphism (with respect to the Zariski topologies).*

Using Halter-Koch's $K-$function rings, it was proven in [Kwegna-Heubo, 2010] and, independently, in [Finocchiaro-Fontana-Loper, 2013b] as a particular case of a more general result the following:

Theorem 2

Let A be any *subring of K, and let*

$$\mathtt{Kr}(K|A) := \bigcap\{V(X) \mid V \in \mathtt{Zar}(K|A)\}.$$

Note that $\mathcal{A} := \mathtt{Kr}(K|A)$ is a $K-$function ring, by F. Halter-Koch's theory.

- *The canonical map $\sigma : \mathtt{Zar}(K|A) \to \mathtt{Zar}(K(X)|\mathcal{A})$, $V \mapsto V(X)$ is an homeomorphism.*

- *The canonical map $\varphi : \mathtt{Zar}(K|A) \cong \mathtt{Zar}(K(X)|\mathcal{A}) \to \mathtt{Spec}(\mathcal{A})$ be the map sending a valuation overring of $\mathcal{A}$ into its center on $\mathcal{A}$, composed with the homeomorphism σ, establishes a homeomorphism (with respect to the Zariski topologies).*

Using Halter-Koch's K–function rings, it was proven in [Kwegna-Heubo, 2010] and, independently, in [Finocchiaro-Fontana-Loper, 2013b] as a particular case of a more general result the following:

Theorem 2

Let A be any *subring of K, and let*

$$\mathtt{Kr}(K|A) := \bigcap\{V(X) \mid V \in \mathtt{Zar}(K|A)\}.$$

Note that $\boldsymbol{\mathcal{A}} := \mathtt{Kr}(K|A)$ is a $K-$function ring, by F. Halter-Koch's theory.

- *The canonical map $\sigma : \mathtt{Zar}(K|A) \to \mathtt{Zar}(K(X)|\boldsymbol{\mathcal{A}})$, $V \mapsto V(X)$ is an homeomorphism.*
- *The canonical map $\varphi : \mathtt{Zar}(K|A) \cong \mathtt{Zar}(K(X)|\boldsymbol{\mathcal{A}}) \to \mathtt{Spec}(\boldsymbol{\mathcal{A}})$ be the map sending a valuation overring of $\boldsymbol{\mathcal{A}}$ into its center on $\boldsymbol{\mathcal{A}}$, composed with the homeomorphism σ, establishes a homeomorphism (with respect to the Zariski topologies).*

Using Halter-Koch's K–function rings, it was proven in [Kwegna-Heubo, 2010] and, independently, in [Finocchiaro-Fontana-Loper, 2013b] as a particular case of a more general result the following:

Theorem 2

Let A be any subring of K, and let

$$\mathtt{Kr}(K|A) := \bigcap \{V(X) \mid V \in \mathtt{Zar}(K|A)\}.$$

Note that $\boldsymbol{\mathcal{A}} := \mathtt{Kr}(K|A)$ is a $K-$function ring, by F. Halter-Koch's theory.

- *The canonical map $\sigma : \mathtt{Zar}(K|A) \to \mathtt{Zar}(K(X)|\boldsymbol{\mathcal{A}})$, $V \mapsto V(X)$ is an homeomorphism.*
- *The canonical map $\varphi : \mathtt{Zar}(K|A) \cong \mathtt{Zar}(K(X)|\boldsymbol{\mathcal{A}}) \to \mathtt{Spec}(\boldsymbol{\mathcal{A}})$ be the map sending a valuation overring of $\boldsymbol{\mathcal{A}}$ into its center on $\boldsymbol{\mathcal{A}}$, composed with the homeomorphism σ, establishes a homeomorphism (with respect to the Zariski topologies).*

Using Halter-Koch's K–function rings, it was proven in [Kwegna-Heubo, 2010] and, independently, in [Finocchiaro-Fontana-Loper, 2013b] as a particular case of a more general result the following:

Theorem 2

Let A be any subring of K, and let

$$\mathtt{Kr}(K|A) := \bigcap\{V(X) \mid V \in \mathtt{Zar}(K|A)\}.$$

Note that $\boldsymbol{\mathcal{A}} := \mathtt{Kr}(K|A)$ is a $K-$function ring, by F. Halter-Koch's theory.

- *The canonical map $\sigma : \mathtt{Zar}(K|A) \to \mathtt{Zar}(K(X)|\boldsymbol{\mathcal{A}})$, $V \mapsto V(X)$ is an homeomorphism.*
- *The canonical map $\varphi : \mathtt{Zar}(K|A) \cong \mathtt{Zar}(K(X)|\boldsymbol{\mathcal{A}}) \to \mathtt{Spec}(\boldsymbol{\mathcal{A}})$ be the map sending a valuation overring of $\boldsymbol{\mathcal{A}}$ into its center on $\boldsymbol{\mathcal{A}}$, composed with the homeomorphism σ, establishes a homeomorphism (with respect to the Zariski topologies).*

§4. Zariski topology on spaces of overrings and spaces of semistar operations

In [B. Olberding, 2010] the author –inspired by Zariski's ideas– considers an extension of the Zariski topology on

- $\overline{\mathtt{Overr}}(A)$ the set of the integrally closed overrings of an integral domain A with quotient field K.

This topology can be easily extended on

- $\mathtt{Overr}(A)$ the set of all the overrings of A and, in particular on
- $\mathtt{OverLoc}(A)$ (resp. $\overline{\mathtt{OverLoc}}(A)$) the set of the local overrings (resp. the set of the local integrally closed overrings) of A

(see also Zariski-Samuel treatise, volume II, page 115).

- More generally, in [Finocchiaro-Spirito, 2014], the authors have further extended the setting where it is natural to consider a Zariski-like topology. In this new setting, the set $\mathtt{Overr}(A)$, $\overline{\mathtt{Overr}}(A)$, $\mathtt{OverLoc}(A)$ $\overline{\mathtt{OverLoc}}(A)$ (and, hence, in particular $\mathtt{Zar}(A)$ and $\mathtt{Spec}(A)$), endowed with their Zariski topology, become in a natural way topological subspaces.

§4. Zariski topology on spaces of overrings and spaces of semistar operations

In [B. Olberding, 2010] the author –inspired by Zariski's ideas– considers an extension of the Zariski topology on

- $\overline{\mathtt{Overr}}(A)$ the set of the integrally closed overrings of an integral domain A with quotient field K.

This topology can be easily extended on

- $\mathtt{Overr}(A)$ the set of all the overrings of A and, in particular on
- $\mathtt{OverLoc}(A)$ (resp. $\overline{\mathtt{OverLoc}}(A)$) the set of the local overrings (resp. the set of the local integrally closed overrings) of A (see also Zariski-Samuel treatise, volume II, page 115).
- More generally, in [Finocchiaro-Spirito, 2014], the authors have further extended the setting where it is natural to consider a Zariski-like topology. In this new setting, the set $\mathtt{Overr}(A)$, $\overline{\mathtt{Overr}}(A)$, $\mathtt{OverLoc}(A)$ $\overline{\mathtt{OverLoc}}(A)$ (and, hence, in particular $\mathtt{Zar}(A)$ and $\mathtt{Spec}(A)$), endowed with their Zariski topology, become in a natural way topological subspaces.

§4. Zariski topology on spaces of overrings and spaces of semistar operations

In [B. Olberding, 2010] the author –inspired by Zariski's ideas– considers an extension of the Zariski topology on

- $\overline{\mathtt{Overr}}(A)$ the set of the integrally closed overrings of an integral domain A with quotient field K.

This topology can be easily extended on

- $\mathtt{Overr}(A)$ the set of all the overrings of A and, in particular on
- $\mathtt{OverLoc}(A)$ (resp. $\overline{\mathtt{OverLoc}}(A)$) the set of the local overrings (resp. the set of the local integrally closed overrings) of A

(see also Zariski-Samuel treatise, volume II, page 115).

- More generally, in [Finocchiaro-Spirito, 2014], the authors have further extended the setting where it is natural to consider a Zariski-like topology. In this new setting, the set $\mathtt{Overr}(A)$, $\overline{\mathtt{Overr}}(A)$, $\mathtt{OverLoc}(A)$ $\overline{\mathtt{OverLoc}}(A)$ (and, hence, in particular $\mathtt{Zar}(A)$ and $\mathtt{Spec}(A)$), endowed with their Zariski topology, become in a natural way topological subspaces.

§4. Zariski topology on spaces of overrings and spaces of semistar operations

In [B. Olberding, 2010] the author –inspired by Zariski's ideas– considers an extension of the Zariski topology on

- $\overline{\texttt{Overr}}(A)$ the set of the integrally closed overrings of an integral domain A with quotient field K.

This topology can be easily extended on

- $\texttt{Overr}(A)$ the set of all the overrings of A and, in particular on
- $\texttt{OverLoc}(A)$ (resp. $\overline{\texttt{OverLoc}}(A)$) the set of the local overrings (resp. the set of the local integrally closed overrings) of A

(see also Zariski-Samuel treatise, volume II, page 115).

- More generally, in [Finocchiaro-Spirito, 2014], the authors have further extended the setting where it is natural to consider a Zariski-like topology.

In this new setting, the set $\texttt{Overr}(A)$, $\overline{\texttt{Overr}}(A)$, $\texttt{OverLoc}(A)$ $\overline{\texttt{OverLoc}}(A)$ (and, hence, in particular $\texttt{Zar}(A)$ and $\texttt{Spec}(A)$), endowed with their Zariski topology, become in a natural way topological subspaces.

§4. Zariski topology on spaces of overrings and spaces of semistar operations

In [B. Olberding, 2010] the author –inspired by Zariski's ideas– considers an extension of the Zariski topology on

- $\overline{\texttt{Overr}}(A)$ the set of the integrally closed overrings of an integral domain A with quotient field K.

This topology can be easily extended on

- $\texttt{Overr}(A)$ the set of all the overrings of A and, in particular on
- $\texttt{OverLoc}(A)$ (resp. $\overline{\texttt{OverLoc}}(A)$) the set of the local overrings (resp. the set of the local integrally closed overrings) of A

(see also Zariski-Samuel treatise, volume II, page 115).

- More generally, in [Finocchiaro-Spirito, 2014], the authors have further extended the setting where it is natural to consider a Zariski-like topology. In this new setting, the set $\texttt{Overr}(A)$, $\overline{\texttt{Overr}}(A)$, $\texttt{OverLoc}(A)$ $\overline{\texttt{OverLoc}}(A)$ (and, hence, in particular $\texttt{Zar}(A)$ and $\texttt{Spec}(A)$), endowed with their Zariski topology, become in a natural way topological subspaces.

Let $\mathtt{SStar}(A)$ be the set of all the semistar operations on an integral domain A with quotient field K. For each nonzero sub-A-module E of K, set

$$\mathcal{U}_E := \{\star \in \mathtt{SStar}(A) \mid 1 \in E^\star\}.$$

The collection $\mathcal{U}_E$, for E varying in the set of nonzero sub-A-modules of K, form a subbasis for the open sets of a topology on $\mathtt{SStar}(A)$, called the *Zariski topology*.

It is easy to see that, for F varying in the set of nonzero finitely generated fractional ideals of A, the collection

$$\mathcal{V}_F := \mathcal{U}_F \cap \mathtt{SStar}_f(A) := \{\star \in \mathtt{SStar}_f(A) \mid 1 \in F^\star\}$$

$$\text{(respectively, } \tilde{\mathcal{V}}_F := \mathcal{U}_F \cap \widetilde{\mathtt{SStar}}(A) := \{\star \in \widetilde{\mathtt{SStar}}(A) \mid 1 \in F^\star\}\text{)}$$

forms a subbasis for the open sets of the induced (Zariski) topology on the set $\mathtt{SStar}_f(A)$ of all the semistar operations of finite type on A (respectively, on the set $\widetilde{\mathtt{SStar}}(A)$ of all the stable semistar operations of finite type on A).

Let $\texttt{SStar}(A)$ be the set of all the semistar operations on an integral domain A with quotient field K. For each nonzero sub-A-module E of K, set

$$\mathcal{U}_E := \{\star \in \texttt{SStar}(A) \mid 1 \in E^\star\}.$$

The collection $\mathcal{U}_E$, for E varying in the set of nonzero sub-A-modules of K, form a subbasis for the open sets of a topology on $\texttt{SStar}(A)$, called the *Zariski topology*.

It is easy to see that, for F varying in the set of nonzero finitely generated fractional ideals of A, the collection

$$\mathcal{V}_F := \mathcal{U}_F \cap \texttt{SStar}_f(A) := \{\star \in \texttt{SStar}_f(A) \mid 1 \in F^\star\}$$

$$\text{(respectively, } \tilde{\mathcal{V}}_F := \mathcal{U}_F \cap \widetilde{\texttt{SStar}}(A) := \{\star \in \widetilde{\texttt{SStar}}(A) \mid 1 \in F^\star\}\text{)}$$

forms a subbasis for the open sets of the induced (Zariski) topology on the set $\texttt{SStar}_f(A)$ of all the semistar operations of finite type on A (respectively, on the set $\widetilde{\texttt{SStar}}(A)$ of all the stable semistar operations of finite type on A).

Let $\mathtt{SStar}(A)$ be the set of all the semistar operations on an integral domain A with quotient field K. For each nonzero sub-A-module E of K, set

$$\mathcal{U}_E := \{\star \in \mathtt{SStar}(A) \mid 1 \in E^\star\}\,.$$

The collection $\mathcal{U}_E$, for E varying in the set of nonzero sub-A-modules of K, form a subbasis for the open sets of a topology on $\mathtt{SStar}(A)$, called the *Zariski topology*.

It is easy to see that, for F varying in the set of nonzero finitely generated fractional ideals of A, the collection

$$\mathcal{V}_F := \mathcal{U}_F \cap \mathtt{SStar}_f(A) := \{\star \in \mathtt{SStar}_f(A) \mid 1 \in F^\star\}$$

$$\text{(respectively, } \tilde{\mathcal{V}}_F := \mathcal{U}_F \cap \widetilde{\mathtt{SStar}}(A) := \{\star \in \widetilde{\mathtt{SStar}}(A) \mid 1 \in F^\star\}\text{)}$$

forms a subbasis for the open sets of the induced (Zariski) topology on the set $\mathtt{SStar}_f(A)$ of all the semistar operations of finite type on A (respectively, on the set $\widetilde{\mathtt{SStar}}(A)$ of all the stable semistar operations of finite type on A).

Let $\texttt{SStar}(A)$ be the set of all the semistar operations on an integral domain A with quotient field K. For each nonzero sub-A-module E of K, set

$$\mathcal{U}_E := \{\star \in \texttt{SStar}(A) \mid 1 \in E^\star\}\,.$$

The collection $\mathcal{U}_E$, for E varying in the set of nonzero sub-A-modules of K, form a subbasis for the open sets of a topology on $\texttt{SStar}(A)$, called the *Zariski topology*.

It is easy to see that, for F varying in the set of nonzero finitely generated fractional ideals of A, the collection

$$\mathcal{V}_F := \mathcal{U}_F \cap \texttt{SStar}_f(A) := \{\star \in \texttt{SStar}_f(A) \mid 1 \in F^\star\}$$

$$\left(\text{respectively, } \tilde{\mathcal{V}}_F := \mathcal{U}_F \cap \widetilde{\texttt{SStar}}(A) := \{\star \in \widetilde{\texttt{SStar}}(A) \mid 1 \in F^\star\}\right)$$

forms a subbasis for the open sets of the induced (Zariski) topology on the set $\texttt{SStar}_f(A)$ of all the semistar operations of finite type on A (respectively, on the set $\widetilde{\texttt{SStar}}(A)$ of all the stable semistar operations of finite type on A).

Let $\mathtt{SStar}(A)$ be the set of all the semistar operations on an integral domain A with quotient field K. For each nonzero sub-A-module E of K, set

$$\mathcal{U}_E := \{\star \in \mathtt{SStar}(A) \mid 1 \in E^\star\}\,.$$

The collection $\mathcal{U}_E$, for E varying in the set of nonzero sub-A-modules of K, form a subbasis for the open sets of a topology on $\mathtt{SStar}(A)$, called the *Zariski topology*.

It is easy to see that, for F varying in the set of nonzero finitely generated fractional ideals of A, the collection

$$\mathcal{V}_F := \mathcal{U}_F \cap \mathtt{SStar}_f(A) := \{\star \in \mathtt{SStar}_f(A) \mid 1 \in F^\star\}$$

$$\text{(respectively, } \tilde{\mathcal{V}}_F := \mathcal{U}_F \cap \widetilde{\mathtt{SStar}}(A) := \{\star \in \widetilde{\mathtt{SStar}}(A) \mid 1 \in F^\star\}\text{)}$$

forms a subbasis for the open sets of the induced (Zariski) topology on the set $\mathtt{SStar}_f(A)$ of all the semistar operations of finite type on A (respectively, on the set $\widetilde{\mathtt{SStar}}(A)$ of all the stable semistar operations of finite type on A).

Let $\mathtt{SStar}(A)$ be the set of all the semistar operations on an integral domain A with quotient field K. For each nonzero sub-A-module E of K, set

$$\mathcal{U}_E := \{\star \in \mathtt{SStar}(A) \mid 1 \in E^\star\}.$$

The collection $\mathcal{U}_E$, for E varying in the set of nonzero sub-A-modules of K, form a subbasis for the open sets of a topology on $\mathtt{SStar}(A)$, called the *Zariski topology*.

It is easy to see that, for F varying in the set of nonzero finitely generated fractional ideals of A, the collection

$$\mathcal{V}_F := \mathcal{U}_F \cap \mathtt{SStar}_f(A) := \{\star \in \mathtt{SStar}_f(A) \mid 1 \in F^\star\}$$

$$\text{(respectively, } \tilde{\mathcal{V}}_F := \mathcal{U}_F \cap \widetilde{\mathtt{SStar}}(A) := \{\star \in \widetilde{\mathtt{SStar}}(A) \mid 1 \in F^\star\}\text{)}$$

forms a subbasis for the open sets of the induced (Zariski) topology on the set $\mathtt{SStar}_f(A)$ of all the semistar operations of finite type on A (respectively, on the set $\widetilde{\mathtt{SStar}}(A)$ of all the stable semistar operations of finite type on A).

Let $\texttt{SStar}(A)$ be the set of all the semistar operations on an integral domain A with quotient field K. For each nonzero sub-A-module E of K, set

$$\mathcal{U}_E := \{\star \in \texttt{SStar}(A) \mid 1 \in E^\star\}.$$

The collection $\mathcal{U}_E$, for E varying in the set of nonzero sub-A-modules of K, form a subbasis for the open sets of a topology on $\texttt{SStar}(A)$, called the *Zariski topology*.

It is easy to see that, for F varying in the set of nonzero finitely generated fractional ideals of A, the collection

$$\mathcal{V}_F := \mathcal{U}_F \cap \texttt{SStar}_f(A) := \{\star \in \texttt{SStar}_f(A) \mid 1 \in F^\star\}$$

$$\text{(respectively, } \tilde{\mathcal{V}}_F := \mathcal{U}_F \cap \widetilde{\texttt{SStar}}(A) := \{\star \in \widetilde{\texttt{SStar}}(A) \mid 1 \in F^\star\}\text{)}$$

forms a subbasis for the open sets of the induced (Zariski) topology on the set $\texttt{SStar}_f(A)$ of all the semistar operations of finite type on A (respectively, on the set $\widetilde{\texttt{SStar}}(A)$ of all the stable semistar operations of finite type on A).

It can be shown that

Lemma 4

- *For each* $\star \in \mathtt{SStar}(A)$, $\mathtt{Cl}(\star) = \{\star' \in \mathtt{SStar}(A) \mid \star' \leq \star\}$.
- *The canonical map* $\mathtt{SStar}(A) \to \mathtt{SStar}_f(A)$, $\star \mapsto \star_f$, *is a continuous retraction.*
- *The canonical map* $\mathtt{Overr}(A) \to \mathtt{SStar}_f(A)$, $B \mapsto \star_{\{B\}}$, *is a topological embedding (where* $\star_{\{B\}} \in \mathtt{SStar}_f(A)$ *is defined by* $E^{\star_{\{B\}}} := EB$, *for each nonzero sub-A-module E of K).*

The map $\mathtt{SStar}_f(A) \to \mathtt{Overr}(A)$, $\star \mapsto A^\star$, *is a continuous retraction.*

It can be shown that

Lemma 4

- *For each* $\star \in \mathtt{SStar}(A)$, $Cl(\star) = \{\star' \in \mathtt{SStar}(A) \mid \star' \leq \star\}$.
- *The canonical map* $\mathtt{SStar}(A) \to \mathtt{SStar}_f(A)$, $\star \mapsto \star_f$, *is a continuous retraction.*
- *The canonical map* $\mathtt{Overr}(A) \to \mathtt{SStar}_f(A)$, $B \mapsto \star_{\{B\}}$, *is a topological embedding (where* $\star_{\{B\}} \in \mathtt{SStar}_f(A)$ *is defined by* $E^{\star_{\{B\}}} := EB$, *for each nonzero sub-A-module E of K).*
  The map $\mathtt{SStar}_f(A) \to \mathtt{Overr}(A)$, $\star \mapsto A^\star$, *is a continuous retraction.*

It can be shown that

Lemma 4

- *For each* $\star \in \mathtt{SStar}(A)$, $Cl(\star) = \{\star' \in \mathtt{SStar}(A) \mid \star' \leq \star\}$.
- *The canonical map* $\mathtt{SStar}(A) \to \mathtt{SStar}_f(A)$, $\star \mapsto \star_f$, *is a continuous retraction.*
- *The canonical map* $\mathtt{Overr}(A) \to \mathtt{SStar}_f(A)$, $B \mapsto \star_{\{B\}}$, *is a topological embedding (where* $\star_{\{B\}} \in \mathtt{SStar}_f(A)$ *is defined by* $E^{\star_{\{B\}}} := EB$, *for each nonzero sub-A-module E of K).* *The map* $\mathtt{SStar}_f(A) \to \mathtt{Overr}(A)$, $\star \mapsto A^\star$, *is a continuous retraction.*

It can be shown that

Lemma 4

- *For each* $\star \in \mathtt{SStar}(A)$, $\mathcal{Cl}(\star) = \{\star' \in \mathtt{SStar}(A) \mid \star' \leq \star\}$.
- *The canonical map* $\mathtt{SStar}(A) \to \mathtt{SStar}_f(A)$, $\star \mapsto \star_f$, *is a continuous retraction.*
- *The canonical map* $\mathtt{Overr}(A) \to \mathtt{SStar}_f(A)$, $B \mapsto \star_{\{B\}}$, *is a topological embedding (where* $\star_{\{B\}} \in \mathtt{SStar}_f(A)$ *is defined by* $E^{\star_{\{B\}}} := EB$, *for each nonzero sub-A-module* E *of* K*).*

The map $\mathtt{SStar}_f(A) \to \mathtt{Overr}(A)$, $\star \mapsto A^\star$, *is a continuous retraction.*

It can be shown that

Lemma 4

- *For each* $\star \in \mathtt{SStar}(A)$, $\mathcal{C}l(\star) = \{\star' \in \mathtt{SStar}(A) \mid \star' \leq \star\}$.
- *The canonical map* $\mathtt{SStar}(A) \to \mathtt{SStar}_f(A)$, $\star \mapsto \star_f$, *is a continuous retraction.*
- *The canonical map* $\mathtt{Overr}(A) \to \mathtt{SStar}_f(A)$, $B \mapsto \star_{\{B\}}$, *is a topological embedding (where* $\star_{\{B\}} \in \mathtt{SStar}_f(A)$ *is defined by* $E^{\star_{\{B\}}} := EB$, *for each nonzero sub-A-module* E *of* K*).*

The map $\mathtt{SStar}_f(A) \to \mathtt{Overr}(A)$, $\star \mapsto A^\star$, *is a continuous retraction.*

We collect in the following theorem some results that can be obtained from the work by [Finocchiaro, 2013], by [C. Finocchiaro-D. Spirito, 2014] and by [C. Finocchiaro-M. Fontana-D. Spirito, in preparation]

Theorem 5

Let A be an integral domain. Then, $\texttt{SStar}_f(A)$, $\widetilde{\texttt{SStar}}(A)$, $\texttt{Overr}(A)$, $\overline{\texttt{Overr}}(A)$, $\texttt{OverLoc}(A)$ *and* $\overline{\texttt{OverLoc}}(A)$, *endowed with their Zariski topologies, are spectral spaces.*

These results were obtained by means of new techniques and, in particular, by means of a characterization, given in [Finocchiaro, 2013], for a topological space to be a spectral space using "appropriate" ultrafilter topologies.

We collect in the following theorem some results that can be obtained from the work by [Finocchiaro, 2013], by [C. Finocchiaro-D. Spirito, 2014] and by [C. Finocchiaro-M. Fontana-D. Spirito, in preparation]

Theorem 5

Let A be an integral domain. Then,
$\mathtt{SStar}_f(A)$, $\widetilde{\mathtt{SStar}}(A)$, $\mathtt{Overr}(A)$, $\overline{\mathtt{Overr}}(A)$, $\mathtt{OverLoc}(A)$ *and* $\overline{\mathtt{OverLoc}}(A)$, *endowed with their Zariski topologies, are spectral spaces.*

These results were obtained by means of new techniques and, in particular, by means of a characterization, given in [Finocchiaro, 2013], for a topological space to be a spectral space using "appropriate" ultrafilter topologies.

We collect in the following theorem some results that can be obtained from the work by [Finocchiaro, 2013], by [C. Finocchiaro-D. Spirito, 2014] and by [C. Finocchiaro-M. Fontana-D. Spirito, in preparation]

Theorem 5

Let A be an integral domain. Then, $\mathtt{SStar}_f(A)$, $\widetilde{\mathtt{SStar}}(A)$, $\mathtt{Overr}(A)$, $\overline{\mathtt{Overr}}(A)$, $\mathtt{OverLoc}(A)$ *and* $\overline{\mathtt{OverLoc}}(A)$, *endowed with their Zariski topologies, are spectral spaces.*

These results were obtained by means of new techniques and, in particular, by means of a characterization, given in [Finocchiaro, 2013], for a topological space to be a spectral space using "appropriate" ultrafilter topologies.

Canonical embeddings of spectral spaces

$$
\begin{tikzcd}
 & \mathtt{SStar}(A) & & \\
 & \mathtt{SStar}_f(A) \arrow[u, hook] & & \\
 & \mathtt{Overr}(A) \arrow[u, hook] & & \widetilde{\mathtt{SStar}}(A) \cong \mathbf{LS_f}(A) \arrow[ull, hook] \\
\overline{\mathtt{Overr}}(A) \arrow[ur, hook] & & \mathtt{OverLoc}(A) \arrow[ul, hook] & \\
 & \overline{\mathtt{OverLoc}}(A) \arrow[ul, hook] \arrow[ur, hook] & & \\
 & \mathtt{Zar}(A) \arrow[u, hook] \arrow[r, two heads] & \mathtt{Spec}(A) \arrow[uu, hook] \arrow[uuur, hook] &
\end{tikzcd}
$$

Let $\boldsymbol{\mathcal{B}}$ denote a non-empty collection of overrings of A and, for any $B \in \boldsymbol{\mathcal{B}}$, let $\star_B$ be a semistar operation on B. An interesting question posed in [Chapman-Glaz, 2000, Problem 44] is the following:

Problem. Find conditions on $\boldsymbol{\mathcal{B}}$ and on the semistar operations $\star_B$ under which the semistar operation $\star_{\boldsymbol{\mathcal{B}}}$ on A defined by $E^{\star_{\boldsymbol{\mathcal{B}}}} := \bigcap\{(EB)^{\star_B} \mid B \in \boldsymbol{\mathcal{B}}\}$, for all $E \in \overline{\boldsymbol{\mathcal{F}}}(A)$, is of finite type.

Note that, *if* $A = \bigcap\{B \mid B \in \boldsymbol{\mathcal{B}}\}$ *is locally finite and each* $\star_B$ *is a star operation on* B *of finite type, then* in [D. D. Anderson, 1988, Theorem 2] the author proved that $\star_{\boldsymbol{\mathcal{B}}}$ *is a star operation on* A *of finite type.*

Let $\boldsymbol{\mathcal{B}}$ denote a non-empty collection of overrings of A and, for any $B \in \boldsymbol{\mathcal{B}}$, let $\star_B$ be a semistar operation on B. An interesting question posed in [Chapman-Glaz, 2000, Problem 44] is the following:

Problem. Find conditions on $\boldsymbol{\mathcal{B}}$ and on the semistar operations $\star_B$ under which the semistar operation $\star_{\boldsymbol{\mathcal{B}}}$ on A defined by
$E^{\star_{\boldsymbol{\mathcal{B}}}} := \bigcap \{(EB)^{\star_B} \mid B \in \boldsymbol{\mathcal{B}}\}$, for all $E \in \overline{\boldsymbol{\mathfrak{F}}}(A)$, is of finite type.

Note that, *if $A = \bigcap\{B \mid B \in \boldsymbol{\mathcal{B}}\}$ is locally finite and each $\star_B$ is a star operation on B of finite type, then* in [D. D. Anderson, 1988, Theorem 2] the author proved that *$\star_{\boldsymbol{\mathcal{B}}}$ is a star operation on A of finite type.*

Let $\boldsymbol{\mathcal{B}}$ denote a non-empty collection of overrings of A and, for any $B \in \boldsymbol{\mathcal{B}}$, let $\star_B$ be a semistar operation on B. An interesting question posed in [Chapman-Glaz, 2000, Problem 44] is the following:

Problem. Find conditions on $\boldsymbol{\mathcal{B}}$ and on the semistar operations $\star_B$ under which the semistar operation $\star_{\boldsymbol{\mathcal{B}}}$ on A defined by
$E^{\star_{\boldsymbol{\mathcal{B}}}} := \bigcap \{(EB)^{\star_B} \mid B \in \boldsymbol{\mathcal{B}}\}$, for all $E \in \overline{\mathfrak{F}}(A)$, is of finite type.

Note that, *if $A = \bigcap\{B \mid B \in \boldsymbol{\mathcal{B}}\}$ is locally finite and each $\star_B$ is a star operation on B of finite type, then* in [D. D. Anderson, 1988, Theorem 2] the author proved that *$\star_{\boldsymbol{\mathcal{B}}}$ is a star operation on A of finite type.*

Through the years, several partial answers to this question were given and they are mainly topological in nature.

- For example, in [Fontana-Huckaba, 2000, Corollary 4.6], a (topological) description of when the semistar operation $\star_{\mathcal{B}}$ is of finite type was given when $\mathcal{B}$ is a family of localizations of A and $\star_B$ is the identity semistar operation on B, for each $B \in \mathcal{B}$.

- More recently, in [Finocchiaro-Fontana-Loper, 2013b], it was proved that *if $\mathcal{B}$ is a quasi-compact subspace in $\mathtt{Zar}(K|A)$ (endowed with the Zariski topology) and $\star_B$ is the identity (semi)star operation on B, for each $B \in \mathcal{B}$, then $\star_{\mathcal{B}}$ is of finite type.*

Through the years, several partial answers to this question were given and they are mainly topological in nature.

• For example, in [Fontana-Huckaba, 2000, Corollary 4.6], a (topological) description of when the semistar operation $\star_{\boldsymbol{\mathcal{B}}}$ is of finite type was given when $\boldsymbol{\mathcal{B}}$ is a family of localizations of A and $\star_B$ is the identity semistar operation on B, for each $B \in \boldsymbol{\mathcal{B}}$.

• More recently, in [Finocchiaro-Fontana-Loper, 2013b], it was proved that *if $\mathcal{B}$ is a quasi-compact subspace in $\mathtt{Zar}(K|A)$ (endowed with the Zariski topology) and $\star_B$ is the identity (semi)star operation on B, for each $B \in \mathcal{B}$, then $\star_{\mathcal{B}}$ is of finite type.*

Through the years, several partial answers to this question were given and they are mainly topological in nature.

- For example, in [Fontana-Huckaba, 2000, Corollary 4.6], a (topological) description of when the semistar operation $\star_{\boldsymbol{\mathcal{B}}}$ is of finite type was given when $\boldsymbol{\mathcal{B}}$ is a family of localizations of A and $\star_B$ is the identity semistar operation on B, for each $B \in \boldsymbol{\mathcal{B}}$.

- More recently, in [Finocchiaro-Fontana-Loper, 2013b], it was proved that *if $\boldsymbol{\mathcal{B}}$ is a quasi-compact subspace in $\mathtt{Zar}(K|A)$ (endowed with the Zariski topology) and $\star_B$ is the identity (semi)star operation on B, for each $B \in \boldsymbol{\mathcal{B}}$, then $\star_{\boldsymbol{\mathcal{B}}}$ is of finite type.*

Another more natural way to see the problem stated above is the following.

Problem. Let $\mathcal{S}$ be any non-empty collection of semistar operations on A and let $\wedge_{\mathcal{S}}$ be the semistar operation defined by $E^{\wedge_{\mathcal{S}}} := \bigcap\{E^{\star} \mid \star \in \mathcal{S}\}$ for all $E \in \overline{\mathbf{F}}(A)$.
Find conditions on the set $\mathcal{S}$ for the semistar operation $\wedge_{\mathcal{S}}$ on A to be of finite type.

Note that it is not so difficult to show that the constructions of the semistar operations of the type $\star_{\mathcal{B}}$ and $\wedge_{\mathcal{S}}$ are essentially equivalent, in the sense that every semistar operation $\star_{\mathcal{B}}$ can be interpreted as one of the type $\wedge_{\mathcal{S}}$, and conversely.

Another more natural way to see the problem stated above is the following.

Problem. Let $\boldsymbol{\mathcal{S}}$ be any non-empty collection of semistar operations on A and let $\wedge_{\boldsymbol{\mathcal{S}}}$ be the semistar operation defined by $E^{\wedge_{\mathcal{S}}} := \bigcap\{E^\star \mid \star \in \boldsymbol{\mathcal{S}}\}$ for all $E \in \overline{\mathbf{F}}(A)$.
Find conditions on the set $\boldsymbol{\mathcal{S}}$ for the semistar operation $\wedge_{\boldsymbol{\mathcal{S}}}$ on A to be of finite type.

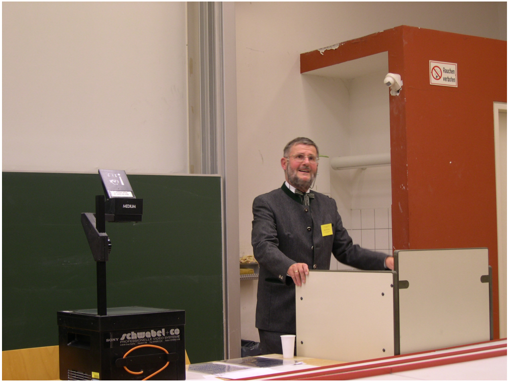
Note that it is not so difficult to show that the constructions of the semistar operations of the type $\star_{\mathcal{B}}$ and $\wedge_{\mathcal{S}}$ are essentially equivalent, in the sense that every semistar operation $\star_{\mathcal{B}}$ can be interpreted as one of the type $\wedge_{\mathcal{S}}$, and conversely.

Another more natural way to see the problem stated above is the following.

Problem. Let $\boldsymbol{\mathcal{S}}$ be any non-empty collection of semistar operations on A and let $\wedge_{\boldsymbol{\mathcal{S}}}$ be the semistar operation defined by $E^{\wedge_{\mathcal{S}}} := \bigcap\{E^\star \mid \star \in \boldsymbol{\mathcal{S}}\}$ for all $E \in \overline{\mathbf{F}}(A)$.
Find conditions on the set $\boldsymbol{\mathcal{S}}$ for the semistar operation $\wedge_{\boldsymbol{\mathcal{S}}}$ on A to be of finite type.

Note that it is not so difficult to show that the constructions of the semistar operations of the type $\star_{\boldsymbol{\mathcal{B}}}$ and $\wedge_{\boldsymbol{\mathcal{S}}}$ are essentially equivalent, in the sense that every semistar operation $\star_{\boldsymbol{\mathcal{B}}}$ can be interpreted as one of the type $\wedge_{\boldsymbol{\mathcal{S}}}$, and conversely.

Thanks!


MEDIUM
schwabel+co
SONY
PROFESSIONELLE VIDEO SYSTEME
Rauchen
verboten

REFERENCES

- D. D. Anderson, Star operations induced by overrings, *Comm. Algebra* **16** (1988), 2535–2553.
- K.E. Aubert, On the ideal theory of commutative semi-groups, *Mathematica Scandinavica* **1** (1953), 39–54.
- K.E. Aubert, Une théorie générale des idéaux. *Séminaire Dubreil: Algèbre et théorie des nombres,* **9** (1955-1956), Exp. No. 8, 12 pp.
- S. Chapman and S. Glaz, One hundred problems in commutative ring theory, in *"Non-Noetherian Commutative Ring Theory", Math. Appl.* **520**, Kluwer Acad. Publ., Dordrecht, 2000, 459–476.
- P.G. Lejeune Dirichlet, *Vorlesungen über Zahlentheorie* (supplements by R. Dedekind), Vieweg 1871.
- David E. Dobbs, Richard Fedder, and Marco Fontana, Abstract Riemann surfaces of integral domains and spectral spaces, *Ann. Mat. Pura Appl.* **148** (1987), 101–115.
- David E. Dobbs and Marco Fontana, Kronecker Function Rings and Abstract Riemann Surfaces, *J. Algebra* **99** (1986), 263–274.
- H.M. Edwards, The genesis of ideal theory, *Arch. Hist. Exact Sci.* **23** (1980/81), 321–378.

- H.M. Edwards, *Divisor theory*, Springer Science & Business Media, 1990.
- C. Finocchiaro, Spectral spaces and ultrafilters, *Comm. Algebra* (2014, to appear).
- C. Finocchiaro, M. Fontana and K. A. Loper, Ultrafilter and constructible topologies on spaces of valuation domains, *Comm. Algebra* **41** (2013), 1825–1835.
- C. Finocchiaro, M. Fontana and K. A. Loper, The constructible topology on the spaces of valuation domains, *Trans. Amer. Math. Soc.* **65** (2013), 6199–6216.
- C. Finocchiaro and D. Spirito, Some topological considerations on semistar operations, *J. Algebra* (2014, to appear).
- Marco Fontana and James Huckaba, Localizing systems and semistar operations, in *"Non-Noetherian commutative ring theory"*, 169–197, *Math. Appl.* **520**, Kluwer Acad. Publ., Dordrecht, 2000.
- M. Fontana and K. A. Loper, Kronecker function rings: a general approach, in *Ideal theoretic methods in commutative algebra* (Columbia, MO, 1999), 189–205, Lecture Notes in Pure and Appl. Math., **220**, Dekker, New York, 2001.
- M. Fontana and K. A. Loper, Nagata rings, Kronecker function rings and related semistar operations, *Comm. Algebra* **31** (2003), 4775–4805.

- Marco Fontana and K. Alan Loper, A generalization of Kronecker function rings and Nagata rings, *Forum Math.* **19** (2007), 971–1004.
- Marco Fontana and K. Alan Loper, The patch topology and the ultrafilter topology on the prime spectrum of a commutative ring, *Comm. Algebra* **36** (2008), 2917–2922.
- Melvin Hochster, Prime ideal structure in commutative rings, *Trans. Amer. Math. Soc.* **142** (1969), 43–60.
- Franz Halter-Koch, *Ideal Systems: An Introduction to Multiplicative Ideal Theory,* M. Dekker, New York, 1998.
- Franz Halter-Koch, Kronecker function rings and generalized integral closures, *Comm. Algebra* **31** (2003), 45–59.
- Franz Halter-Koch, Lorenzen Monoids: A multiplicative approach to Kronecker function rings, *Comm. Algebra* **43** (2015, to appear).
- Paul Jaffard, *Les systèmes d'idéaux,* Dunod, Paris, 1960.
- Leopold Kronecker, Grundzüge einer arithmetischen theorie der algebraischen grössen, *Journal für die reine und angewandte Mathematik* **92** (1882), 1–122.
- W. Krull, Beiträge zur Arithmetik kommutativer Integritätsbereiche, I - II, *Math. Z.* **41** (1936), 545–577; 665–679.

- Franz-Viktor Kuhlmann, Places of algebraic fields in arbitrary characteristic, *Advances Math.* **188** (2004), 399–424.
- E.E. Kummer, Über die Zerlegung der aus Wurzeln der Einheit gebildeten complexen Zahlen in ihre Primfactoren, *J. reine angew. Math.* **35** (1847), 327–367.
- Olivier Kwegna-Heubo, Kronecker function rings of transcendental field extensions, *Comm. Algebra* **38** (2010), 2701–2719.
- P. Lorenzen, Abstrakte Begründung der multiplikativen Idealtheorie, *Mathematische Zeitschrift* **45** (1939), 533–553.
- B. Olberding, Noetherian spaces of integrally closed rings with an application to intersections of valuation rings, *Comm. Algebra* **38** (2010), 3318–3332.
- B. Olberding, Affine schemes and topological closures in the ZariskiRiemann space of valuation rings. *J. Pure Appl. Algebra* (2014), to appear.
- H. Prüfer, Untersuchungen über Teilbarkeitseigenschaften in Körpern, *Journal für die reine und angewandte Mathematik*, **168** (1932), 1–36.
- J.-P. Soublin, Préhistoire des idéaux. *Cahiers du séminaire d'histoire des mathématiques,* **5** (1984), 13–20.
- Niels Schwartz, Sheaves of Abelian l-groups, *Order*, **30** (2013), 497–526.

- Niels Schwartz and Marcus Tressl, Elementary properties of minimal and maximal points in Zariski spectra, *J. Algebra* **323** (2010), 698–728.
- H. Weyl, *Algebraic theory of numbers,* Princeton University Press, 1940.
- Oscar Zariski, The reduction of singularities of an algebraic surface, *Ann. Math.* **40** (1939), 639–689.
- Oscar Zariski, The compactness of the Riemann manifold of an abstract field of algebraic functions, *Bull. Amer. Math. Soc.* **50** (1944), 683–691.
- Oscar Zariski, Reduction of the singularities of algebraic three dimensional varieties, *Ann. Math.* **45** (1944), 472–542.
- Oscar Zariski and Pierre Samuel, *Commutative Algebra, Volume 2,* Springer Verlag, Graduate Texts in Mathematics **29**, New York, 1975 (First Edition, Van Nostrand, Princeton, 1960).