From simple to less simple

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Although in a commutative ring the additive structure is axiomatically stronger than the multiplicative structure, the last one seems to be more important in several cases.
An exemplification of that is what happens in semigroup rings with coefficients in a field, where the semigroup has a big role in the study of the ring. In case of numerical semigroups and numerical semigroup rings this connection is particularly sharp.

It's not surprising that there is a strict relation between the ring $K[t^2, t^3]$ and the semigroup $\langle 2, 3 \rangle$, if K is a field.

For any (additive) semigroup S, the semigroup ring

$$K[S] := \{\mathbf{t}^s; s \in S\} \quad \text{where } \mathbf{t}^a\mathbf{t}^b = \mathbf{t}^{a+b}$$

has several properties depending just on its multiplicative structure, that is the additive structure of S.
To give a concrete simple example, if we take

$$R := K[\mathbb{N}^2] = K[X, Y]$$

the monomial ideals which are not intersection of two proper monomial overideals, i.e. the monomially irreducible ideals are those of the form

$$(X^i), \quad (Y^j), \quad (X^i, Y^j)$$

However

Proposition

If a monomial ideal of $K[X, Y]$ is monomially irreducible, then it is irreducible (i.e. it is not the intersection of two proper over ideals, even if nonmonomial ideals are allowed).

Thus the semigroup $\mathbb{N}^2$ says everything about the decomposition of a monomial ideal into irreducible ideals. This can be generalized. If S is an affine semigroup (i.e. $S \subseteq \mathbb{Z}^d$ is finitely generated) and F is a face of S, consider:

$$F - S$$

In case $S = \mathbb{N}^2$ we have the faces

$$F_0 = \{(0,0)\}, \ F_1 = \{(n,0); n \in \mathbb{N}\}, \ \ F_2 = \{(0,m); m \in \mathbb{N}\}, \ \ \mathbb{N}^2$$

It turns out that a monomial ideal I of $K[S]$ is irreducible if and only if the monomials in I are in

$$S \setminus (a + F - S)$$

for some $a \in \mathbb{Z}^d$.

Thus in the example $K[\mathbb{N}^2] = K[X, Y]$ we get again the ideals

$$(X^i), \quad (Y^j), \quad (X^i, Y^j)$$

Not only for semigroup rings the multiplicative structure of the ring gives important information.
The description of the multiplicative structure of an integral domain is often enough to deduce important properties of the domain.
Franz Halter-Koch writes in the preface of Ideal Systems “In this volume we adhere to the philosophy that results that are concerned only with the multiplicative structure should be derived as far as possible without making reference to the additive structure".
So, starting long ago with W. Krull and H. Prüfer a rich theory written in terms of commutative monoids has developed and Graz’s school has had a big role in that.

Look at one of the simplest commutative monoid: a numerical semigroup S, i.e. an additive submonoid of $\mathbb{N}$ with zero and finite complement in $\mathbb{N}$.

I don't want to talk now of the semigroup rings $K[S]$, but I would like to point out how several easy and natural concepts coming from numerical semigroups have been generalized to rings, which seem to have nothing to do with semigroup rings.

One of this is thealmost Gorenstein property which comes from the almost symmetric property in numerical semigroups.

Let's start with the num. semigroup context. S is a *numerical semigroup*.

$M = S \setminus \{0\}$ is the *maximal ideal* of S

g is the *Frobenius number* of S, that is the greatest integer which does not belong to S.

A *relative ideal* of S is a nonempty subset I of $\mathbb{Z}$ (which is the quotient group of S) such that $I + S \subseteq I$ and $I + s \subseteq S$, for some $s \in S$.

If I, J are relative ideals of S, then the following is a relative ideals too:

$$I - J = \{z \in \mathbb{Z} \mid z + J \subseteq I\}$$

A particular relative ideal of S plays a special role. It is the *canonical ideal*

$$\Omega = \{g - x \mid x \in \mathbb{Z} \setminus S\}$$

So

$$x \notin S \Rightarrow g - x \in \Omega$$

$$x \in S \Rightarrow g - x \notin \Omega$$

Of course it is $S \subseteq \Omega \subseteq \mathbb{N}$

($s \in S$ then $x = g - s \notin S$, so $s = g - (g - s) = g - x \in \Omega$).

Example

$$S = \{0, 4, 7, 8, 11, 12, \rightarrow\}$$

Frobenius number $g = 10$

$$0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$$

x is *symmetric* to $g - x = 10 - x$
0 is symmetric to 10
1 is symmetric to 9
2 is symmetric to 8 ...

$$\Omega = \{\text{black}\} \cup \{\text{red}\}$$

Facts:

- The dual of an ideal I of S is

$$(\Omega - I) = \{g - x \mid x \in \mathbb{Z} \setminus I\}$$

- $\Omega - (\Omega - I) = I$, for each ideal I of S.
- Let Ω' be a relative ideal of S. Then

$$\Omega' - (\Omega' - I) = I$$

  for each ideal I of S if and only if $\Omega' = z + \Omega$, for some $z \in \mathbb{Z}$.
- The minimal number of generators of Ω is the *type* t of the semigroup S (if $t = 1$ or equivalently $\Omega = S$, then the numerical semigroup S is classically called *symmetric*).

S is *almost symmetric* if $\Omega \subseteq M - M$

Example

$$S = \langle 5, 8, 9, 12 \rangle = \{0, 5, 8, 9, 10, 12, \rightarrow\}$$

$$0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, \rightarrow$$

is almost symmetric because

$$\Omega = \{0, 4, 5, 7, 8, 9, 10, 12, \rightarrow\} = \langle 0, 4, 7 \rangle$$

$$4, 7 \in M - M = \{0, 4, 5, 7, \rightarrow\}$$

Let (R, m) be a local Cohen Macaulay ring possessing a canonical module ω.

R has a canonical module if and only if $R \cong G/I$ for some Gorenstein ring G. In particular any quotient of a regular ring has a canonical module.

If $\dim R = 1$, then a canonical module is a fractional regular ideal ω of R such that $\omega : (\omega : I) = I$, for all fractional regular ideals I of R.

If $\dim R > 1$, then $\mathrm{Hom}_R(-, \omega)$ is a dualizing functor on the category of maximal Cohen Macaulay modules and we have

$$\mathrm{Hom}_R(\mathrm{Hom}_R(M, \omega), \omega) \cong M$$

for each maximal CM module M.

Among CM local local rings R having a canonical module there is a very special class. If the canonical module exists and is isomorphic to R, then R is a *Gorenstein ring*. So in dimension one, R is Gorenstein if and only if each ideal I is divisorial:

$$\omega : (\omega : I) = R : (R : I) = I$$

If (R, m) is a local one-dimensional CM ring with finite integral closure (i.e. *analytically unramified*), it has a canonical ideal ω which can be supposed

$$R \subseteq \omega \subseteq \overline{R}$$

In Journal of Algebra (1997), besides *almost symmetric numerical semigroups* , we (V.B. - R. Fröberg) defined a one-dimensional analytically unramified ring R to be *almost Gorenstein* if one of the equivalent conditions are satisfied

1. $\omega \subseteq m : m$
   (we have $\Omega \subseteq M - M$ for almost symm. semigroups)
2. $m = \omega m$
3. $\omega : m = m : m$ (supposing R not a DVR)
4. $m\omega \subset R$

R is almost Gorenstein if and only if $m\omega \subset R$.
Thus for a one-dimensional analytically unramified ring we have an exact sequence of R-modules

$$0 \to R \to \omega \to \omega/R \to 0$$

and R is almost Gorenstein if and only $m\omega \subset R$, i.e. if and only if $m\,(\omega/R) = 0$.
Thus R is almost Gorenstein if and only if there is an exact sequence of R-modules

$$0 \to R \to \omega \to C \to 0$$

such that $mC = 0$
In particular, if R is Gorenstein, $R = \omega$ and $C = 0$.

S. Goto - N. Matsuoka - T. Thi Phuong. Almost Gorenstein rings. (J. Algebra 2013) extended this definition to any one-dimensional CM ring which has a canonical ideal (I talked of that in Bressanone). If the residue field is finite not always it's possible to have ω between R and its integral closure, so they prefer to talk of the canonical ideal as an m-primary ideal.

More recently S. Goto, R. Takahashi and N. Taniguchi (arXiv 1403.3599, to appear in J. Pure Appl. Alg.) extended this definition to local CM rings of any dimension which have a canonical module.

The smart generalized definition of almost Gorenstein rings given in that paper has a form similar to our definition in case of a one-dimensional analytically unramified ring.

In fact a Cohen-Macaulay local ring (R, m) of any Krull dimension d, possessing a canonical module ω is defined almost Gorenstein if there is an exact sequence of R-modules

$$0 \to R \to \omega \to C \to 0$$

such that $\mu_R(C) = e_m^0(C)$, where $\mu_R(C) = \ell_R(C/mC)$ is the number of generators of C and $e_m^0(C)$ is the multiplicity of C with respect to m.

It turns out that $\dim_R C = d - 1$.
This condition generalizes the condition $mC = 0$. In fact:

$$mC = 0 \quad \text{for} \quad \dim R = 1$$

becomes

$$mC = (f_1, \ldots, f_{d-1})C \quad \text{for} \quad \dim R = d$$

for some $f_1, \ldots, f_{d-1} \in m$ i.e.

$$\mu_R(C) = \ell(C/mC) = \ell(C/(f_1, \ldots, f_{d-1})C = e^0_m(C)$$

Since in general, for an R-module C, $\mu_R(C) \leq e^0_m(C)$, in the definition above C is requested to be maximally generated.

Lemma (S. Goto, R. Takahashi, N. Taniguchi)

If $f \in R$ is a nonzerodivisor and if $R/(f)$ is almost Gorenstein, then R is almost Gorenstein too.

Example Let

$$R = K[[X_1, X_2, X_3, Y_1, Y_2, Y_3]]/I_2(\mathbb{M})$$

where $I_2(\mathbb{M})$ is the ideal generated by the 2×2 minors of the matrix

$$\mathbb{M} = \begin{pmatrix} X_1 & X_2 & X_3 \\ Y_1 & Y_2 & Y_3 \end{pmatrix}.$$

It is well known that R is a CM local ring of dimension 4. Since $X_1 - Y_3, X_2 - Y_1, X_3 - Y_2$ is a regular sequence in R and factoring out these elements we get

$$T = R/(X_1 - Y_3, X_2 - Y_1, X_3 - Y_2)$$

which is an almost Gorenstein CM ring of dimension one, by the Lemma R is almost Gorenstein.

Among several results they prove that the notion is stable by idealization. More precisely:
If E is an R-module, the Nagata idealization of E over R, $R \ltimes E$, is $R \oplus E$, with the product defined:

$$(r, i)(s, j) = (rs, rj + si)$$

$R \ltimes E$ is an extension of R containing an ideal $\bar{E}$ such that $\bar{E}^2 = 0$. So $R \ltimes E$ is not a reduced ring.

Theorem (S. Goto, N. Matsuoka, T. Phuong, + S. Goto, R. Takahashi, N. Taniguchi)

Let (R, m) be a CM local ring OF ANY DIMENSION . Then the following conditions are equivalent:
(1) *$R \ltimes m$ is an almost Gorenstein ring*
(2) *R is an almost Gorenstein ring*

It's a way to construct many examples of analytically ramified almost Gorenstein rings, that are not Gorenstein.

Marco and Marco (D'Anna and Fontana) introduced the amalgamated duplication of a ring R along an R-module E, $R \bowtie E$ as $R \oplus E$, with the product defined:

$$(r, i)(s, j) = (rs, rj + si + ij)$$

$R \bowtie E$ can be reduced and it is always reduced if R ia a domain.

In Bressanone's talk I asked the following (for one-dimensional rings):
Question Are the following equivalent ??
(1) $R \bowtie m$ is an almost Gorenstein ring
(2) R is an almost Gorenstein ring

Let R be a commutative ring with unity and let I be a proper ideal of R. In a joint paper with M. D'Anna and F. Strazzanti we studied the family of quotient rings

$$R(I)_{a,b} = \mathcal{R}_+/(I^2(t^2 + at + b)),$$

where $\mathcal{R}_+$ is the Rees algebra associated to the ring R with respect to I (i.e. $\mathcal{R}_+ = \bigoplus_{n\geq 0} I^n t^n$) and $(I^2(t^2 + at + b))$ is the contraction to $\mathcal{R}_+$ of the ideal generated by $t^2 + at + b$ in $R[t]$.

Thus

$$R(I)_{a,b} = \{\overline{r + it}; r \in R,\ i \in I\}$$

and the product

$$(\overline{r + it})(\overline{s + jt}) = \overline{rs + (rj + si)t + ijt^2}$$

is defined mod the ideal $I^2(t^2 + at + b)$. So, if $t^2 + at + b = t^2$, we get the idealization: $\mathcal{R}_+/(I^2 t^2) \cong R \ltimes I$. In fact

$$(\overline{r + it})(\overline{s + jt}) = \overline{rs + (rj + si)t}$$

If $t^2 + at + b = t^2 - t$, we get the duplication: $\mathcal{R}_+/(I^2(t^2 - t)) \cong R \bowtie I$. In fact:

$$(\overline{r + it})(\overline{s + jt}) = \overline{rs + (rj + si + ij)t}$$

We proved:

- The ring extensions $R \subseteq R(I)_{a,b} \subseteq R[t]/(f(t))$ are both integral and the three rings have the same Krull dimension and the last two have the same total ring of fractions and the same integral closure;
- R is a Noetherian ring if and only if $R(I)_{a,b}$ is a Noetherian ring for all $a, b \in R$ if and only if $R(I)_{a,b}$ is a Noetherian ring for some $a, b \in R$;
- $R = (R, \mathfrak{m})$ is local if and only if $R(I)_{a,b}$ is local. In this case the maximal ideal of $R(I)_{a,b}$ is $\mathfrak{m} \oplus I$ (as R-module), i.e. $\{m + it; m \in \mathfrak{m}, i \in I\}$
- Assume that R is a local CM ring of dimension d. Then $R(I)_{a,b}$ is CM if and only if I is a CM R-module of dimension d. In particular, the Cohen-Macaulayness of $R(I)_{a,b}$ depends only on the ideal I.

- If R is a CM one-dimensional local ring, the CM type of $R(I)_{a,b}$ is

$$t(R(I)_{a,b}) = \ell_R(\frac{(I:I) \cap (R:\mathfrak{m})}{R}) + \ell_R(\frac{I:\mathfrak{m}}{I});$$

  in particular, it does not depend on a and b.
- If R is a CM one-dimensional local ring then $R(I)_{a,b}$ is Gorenstein if and only if I is a canonical ideal.

Go back now to the almost Gorenstein property.

Theorem (V. B., M. D'Anna, F. Strazzanti)

Let (R, m) be a one-dimensional CM local ring. Then the following conditions are equivalent:
(1) $R(m)_{a,b}$ *is an almost Gorenstein ring*
(2) R *is an almost Gorenstein ring*

Indeed this is a corollary of the following

Theorem (V. B., M. D'Anna, F. Strazzanti)

Let (R, m) be a one-dimensional CM local ring with canonical ideal ω and let I be a proper ideal. If z is a minimal reduction of $(\omega : I)$, then $R(I)_{a,b}$ is almost Gorenstein if and only if:
(1) $I(\omega : I) = Iz$
(2) $m(\omega : I) = mz$

In the proof $\omega_{R(I)_{a,b}}$ has a big role . It is known that for local and finite extensions of rings of same dimension:

$$\omega_{R(I)_{a,b}} \cong \mathrm{Hom}_R(R(I)_{a,b}, \omega_R)$$

This also has an easy version for numerical semigroups:
If $S \subset T$ are semigroups with Frobenius numbers g_S, g_T and canonical ideals Ω_S, Ω_T respectively. Then

$$\Omega_T + (g_S - g_T) = \{g_T - x; x \in \mathbb{Z} \setminus T\} + (g_S - g_T) =$$
$$= \{g_S - x; x \in \mathbb{Z} \setminus T\} = \Omega_S - T$$

So, the canonical ideal of T is the dual of T, with respect to Ω_S. Back to the proof: as an R-module

$$\omega_{R(I)_{a,b}} \cong \mathrm{Hom}_R(R(I)_{a,b}, \omega_R) \cong \mathrm{Hom}_R(R \oplus I, \omega_R) \cong$$

$$\cong \mathrm{Hom}_R(R, \omega_R) \oplus \mathrm{Hom}_R(I, \omega_R) \cong \omega_R \oplus (\omega_R : I) \cong \frac{1}{z}(\omega_R : I) \oplus \frac{1}{z}\omega_R.$$

where z is a minimal reduction of $(\omega_R : I)$. We show that

$$\omega_{R(I)_{a,b}} \cong \{\frac{x}{z} + \frac{y}{z}t \mid x \in (\omega_R : I), y \in \omega_R\}$$

with the $R(I)_{a,b}$-module structure given by:

$$(r + it)\left(\frac{x}{z} + \frac{y}{z}t\right) = \left(\frac{rx}{z} - \frac{biy}{z} + \left(\frac{ry}{z} + \frac{ix}{z} - \frac{aiy}{z}\right)t\right) \in K;$$

for $(r + it) \in R(I)_{a,b}$

Let's see how from the second Theorem follows the first one.
Proof. The conditions of the second Theorem are

(1) $I(\omega : I) = Iz$

(2) $m(\omega : I) = mz$

For $I = m$ they become:

(1) $m(\omega : m) = mz$

(2) $m(\omega : m) = mz$

Thus: if R is almost Gorenstein (i.e. $(\omega : m) = (m : m)$), then condition (1) $\equiv$ (2) coincide and are trivially satisfied because a minimal reduction of $(\omega : m) = (m : m)$ is $z = 1$.
Conversely, if $R(m)_{a,b}$ is Gorenstein, then m is a canonical ideal of R, thus $m : m = m : (m : R) = R$ and R is a DVR.
If $R(m)_{a,b}$ is almost Gorenstein but not Gorenstein, since $z = 1$ is a minimal reduction of $\omega : m$ we get $(\omega : m) \subseteq (m : m)$ and so the equality.

Let's see how the second theorem has a formulation for numerical semigroups.

Let S be a numerical semigroup, $I \subset S$ an ideal and $b \in S$ an odd integer. Define the duplication of S with respect to I as

$$S \bowtie^b I := (2 \cdot S) \cup (2 \cdot I + b)$$

where $2 \cdot S = \{2s;\ s \in S\}$

Example

$$S = \{0, 2, 4, 5, \rightarrow\} \qquad I = \{4, 6, 7, \rightarrow\} \quad b = 5$$

$$2 \cdot S = \{0, 4, 8, 10, 12, ..., 2k, ...\} \qquad 2 \cdot I = \{8, 12, 14, ...2k, ...\}$$

$$2 \cdot I + 5 = \{13, 17, 19, 21, ..., 2k + 1, ...\}$$

Thus

$$S \bowtie^5 I = \{0, 4, 8, 9, 10, 12, 13, \rightarrow\}$$

$$S \bowtie^5 I = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, \rightarrow\}$$

Theorem (M. D'Anna - F. Strazzanti)

Let S be a numerical semigroup, $I \subset S$ an ideal, let z be the minimal element of $\Omega - I$ and $b \in S$ an odd integer.
Then $S \bowtie^b I$ is almost symmetric if and only if:
(1) $I + (\Omega - I) = I + z$ *(for rings it was* $I(\omega : I) = Iz$*)*
(2) $M + (\Omega - I) = M + z$ *(for rings it was* $m(\omega : I) = mz$*)*

Condition(1):
$I + (\Omega - I) = I + z \Leftrightarrow (\Omega - I) - z$ is a numerical semigroup

condition (2):
$M+(\Omega-I) = M+z \Leftrightarrow M+(\Omega-I) \subseteq M+z \Leftrightarrow (\Omega-I)-z \subseteq M-M$

Thus we get with duplication an almost symmetric semigroup if we use (mod a translation) $I = \Omega - (\Omega - I) = \Omega - T$, i.e. the dual of an oversemigroup T of S contained in $M - M$.

Procedure:

- choose a semigroup T, $S \subset T \subseteq M - M$
- consider the dual $\Omega - T$ and translate it to the right inside S, obtaining an integral ideal I of S.
- For each odd integer $b \in S$ the semigroup $S \bowtie^b I$ is almost symmetric.

THE END